18. *A Note on Strongly Regular Rings*

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Let R be a ring. We recall Drazin's definitions [2]: an element x of R is called semi-π-regular in R if a positive integer $s=s(x)$ and an element $g=g(x)$ of R exist satisfying either $x^s=xgx^s$ or $x^s=x^sgx$, R being itself called semi-π-regular if every element of R is semi-π-regular in R. An element x of R is called strongly regular in R if an element $a=a(x)$ of R exists satisfying $x=x^2a$. The ring R is itself called strongly regular if every element of R is strongly regular in R. It should be noted that in a strongly regular ring R, $x=x^2a$ if and only if $x=ax^2$ (see [2]).

A subring M of the ring R, following Steinfeld [7], is said to be a quasi-ideal if $RM\cap MR\subseteq M$.

The following result has been proved (see [2, 3, 5]):

Proposition. For an arbitrary ring R the following conditions are equivalent:

(1) R is strongly regular;

(2) R is semi-π-regular and isomorphic to a subdirect sum of division rings;

(3) R is semi-π-regular and contains no non-zero nilpotent elements;

(4) every quasi-ideal M of R is idempotent, i.e. $M^2=M$;

(5) for every right ideal J and every left ideal L of R, $JL=J\cap L\subseteq LJ$ holds.

The concept of group membership in rings was introduced by Ranum [6]. An element a of the ring R is said to be a group member in R if a is contained in a subgroup of R with respect to multiplication. Evidently, the zero element of R is a group member in R. The purpose of this note is to give a characterization of strongly regular rings R in terms of group membership in R.

Namely, we prove the following theorem:

Theorem. For an arbitrary ring R, each of the five conditions (1)–(5) in the previous proposition is equivalent to the statement:

(6) each element of R is a group member of R.

Proof. By virtue of the previous proposition, we need only to show the equivalence of the conditions (1) and (6).

(6) implies (1): Let x be an element in R. Then x is contained in a multiplicative subgroup G of R and hence the equation $x=x^2a$

is solvable for a in G, so x is strongly regular in R.

(1) implies (6): If R is strongly regular and $x \neq 0$ is an element of R with $x = x^2a = ax^2$. Let S be the semigroup generated by x and xa. Clearly, since $ax = a(x^2a) = (ax^2)a = xa$, S consists of all elements of R of the forms xa^n and x^m, $n, m = 1, 2, 3, \cdots$. S contains identity element xa, since

$$x(xa) = x^2a = x,$$
$$x^m(xa) = x^{m-1}(x^2a) = x^{m-1}x = x^m, \text{ for } m = 2, 3, 4, \cdots,$$

and

$$(xa^n)(xa) = (x^2a)a^n = xa^n, \text{ for } n = 1, 2, 3, \cdots.$$

Moreover, each element of S has inverse relative to the identity element xa. In fact,

$$(xa)(xa) = (xa)^2 = xa,$$
$$(xa^n)x^{n-1} = x^na^n = (xa)^n = xa, \text{ for } n = 2, 3, 4, \cdots,$$

and

$$x^m(xa^{m+1}) = x^{m+1}a^{m+1} = xa, \text{ for } m = 1, 2, 3, \cdots.$$

Thus S is a group containing x and hence x is a group member in R. This completes the proof.

It is known (see [4]) that if x is a group member in a ring R then there is a maximal multiplicative subgroup $M(x)$ of R containing x, whose identity element is the identity element of every multiplicative subgroup containing x. Distinct maximal multiplicative subgroups of R are disjoint. This follows that a ring R in which each element is a group member is a class sum of mutually disjoint multiplicative groups. Clifford [1] has proved that a semigroup R is the class sum of mutually disjoint groups if and only if it admits relative inverse, i.e. to each element a of R there exist elements e and a' of R such that $ea = ae = a$ and $aa' = a'a = e$. Therefore, we have the following corollary:

Corollary. For an arbitrary ring R each of the six conditions (1)–(6) is equivalent to the statement that R is a semigroup admitting relative inverse with respect to multiplication.

References

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