17. *On Adjoint Maps between Dual Systems*

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(Comm. by Kinjirô KUNUGI, M.J.A., Feb. 12, 1964)

Let E, F be linear spaces, α a bilinear form on $E\times F$. A pair of E, F is called a *dual system* on α if

1) $\alpha(x, y)=0$ for all $x\in E$ implies $y=0$,

2) $\alpha(x, y)=0$ for all $y\in F$ implies $x=0$.

Let E, F be a dual system on α, and let G, H be a dual system on β. If u is a linear map from E to G, then $\beta(u(x), y')$ is bilinear on $G\times H$, where $y'\in H$. Put $\alpha(x, u^*(y'))=\beta(u(x), y')$ for all $x\in E$ and $y'\in H$, if u^* is defined, it is a map from H to F. u^* is called the *algebraic adjoint* map of u. If B is of finite dimension, u^* is always well defined. In a Hilbert space with many inner products, we can define the adjoint on these inner products of a continuous linear map. The other example is the usual adjoint map. Let A be a subset of E (or F), the *orthogonal part* $A^\perp$ is defined by the set $\{y\,|\,\alpha(x, y)=0$ for all $x\in A\}$ (or $\{x\,|\,\alpha(x, y)=0$ for all $y\in A$). Similarly we can also consider the orthogonal part for G, H, and β.

We shall consider a system in which the adjoint map is well defined.

Then we have the following fundamental

Proposition 1. *For a linear subspace A of E,*

$$u^{*-1}(A^\perp)=(u(A))^\perp.$$

Proposition 2. *For a linear subspace B of G,*

$$(u^{-1}(B))^\perp=u^*(B^\perp).$$

Proof of Proposition 1. Let $y'\in u^{*-1}(A^\perp)$, then for all $x\in A$, $0=\alpha(x, u^*(y'))=\beta(u(x), y')$ and so $y'\in u(A)^\perp$. Conversely if $y'\in(u(A))^\perp$, then $0=\beta(u(x), y')=\alpha(x, u^*(y'))$ for all $x\in A$. Hence $u^*(y')\subset A^\perp$, and $y'\in u^{*-1}(A^\perp)$.

Proof of Proposition 2. Let $y\in u^*(B^\perp)$, there is an element $y'\in B^\perp$ such that $u(y')=y$. For any $x\in u^{-1}(B)$, we have $\alpha(x, y)=\alpha(x, u^*(y'))$ $=\beta(u(x), y')=0$. Hence $y\in(u^{-1}(B))^\perp$.

To prove the converse, we shall consider some linear subspaces. From $u^{-1}(B)\supset\ker(u)$, there is the linear subspace E_1 such that $u^{-1}(B)=\ker(u)\oplus E_1$. Hence $B\subset u(E_1)$. Further there is the linear subspace E_2 such that $E=\ker(u)\oplus E_1\oplus E_2$. Let $\tilde{u}$ be the restriction of u on $E_1\oplus E_2$, then u gives an isomorphism from $E_1\oplus E_2$ to $\mathrm{Im}(u)$. If $G_1=\tilde{u}(E_1)$, $G_2=\tilde{u}(E_2)$, then $B\subset G_1$. Let p be the projection from G to G_1.

For $y \in (u^{-1}(B))^{\perp}$, we define y' by

$$\beta(x', y') = \alpha(\tilde{u}^{-1}(p(x')), y) \quad \text{for } x' \in G.$$

If $x' \in G_1 \oplus G_2$, then $\beta(x', y') = \alpha(\tilde{u}^{-1}(x'), y)$. Hence $\alpha(x, y) = \beta(u(x), y') = \alpha(x, u^*(y'))$ for all $x \in E$. Therefore we have $y = u^*(y')$. Then for $x' \in G$,

$$\beta(x', y') = \alpha(\tilde{u}^{-1}(p(x')),\ u^*(y'))$$
$$= \beta(u(\tilde{u}^{-1}(p(x'))), y') = \beta(p(x'), y').$$

This shows $\beta(x'_3, y') = 0$ for $x'_3 \in G_3$. For each $x' \in B$, we have $x' = x'_1 + x'_3$, $x'_1 \in G_1$, $x'_3 \in G_2$. Let $u(x_1) = x'_1$, then

$$0 = \alpha(x_1, y) = \alpha(x_1, u^*(y'))$$
$$= \beta(u(x), y') = \beta(x'_1, y').$$

Therefore we have

$$\beta(x', y') = \beta(x'_1 + x'_3, y') = \beta(x'_1, y') + \beta(x'_3, y') = 0.$$

This shows $y = u^*(y')$, $y' \in B^{\perp}$, which is equivalent to $y \in u^*(B^{\perp})$.

COROLLARY. $\mathrm{Ker}(u^*) = (\mathrm{Im}(u))^{\perp}$, $\mathrm{Im}(u^*) = (\mathrm{Ker}(u))^{\perp}$.

Proposition 3. $\mathrm{Im}(u) = G$ *if and only if* u^{*-1} *exists.*

Proof. By Corollary $\mathrm{Ker}(u^*) = (\mathrm{Im}(u))^{\perp} = (0)$ if and only if $\mathrm{Im}(u) = G$. This is equivalent to the existence of the inverse of u^*.

Proposition 4. $\mathrm{Im}(u^*) = F$ *if and only if* u^{-1} *exists.*

Proof. u^{-1} exists, if and only if $\mathrm{Ker}(u) = 0$. Hence by Corollary, this means $\mathrm{Im}(u^*) = F$.

If $\mathrm{Ker}(u)$ is of finite dimension, u is called an *α-map*, on the other hand, if $\mathrm{Im}(u)$ is of finite co-dimension, u is called a *β-map*. (See A. Deprit [1].) Then, by Corollary,

Proposition 5. Let u be a linear map from E to F. u is an α-map (or β-map) if and only if u^* is a β-map (or an α-map).

In this case, we have $\dim(\ker(u)) = \mathrm{cod}(\mathrm{Im}(u^*))$ or $\mathrm{cod}(\mathrm{Im}(u)) = \dim(\ker(u^*))$.

Reference

[1] A. Deprit: Contribution à l'étude de l'Algèbre des applications linéaires continues d'un espace localement convexe séparé. Acad. Roy. Belgique, Cl. Sci. Mém. Coll., **8**(31), 1–170 (1959).