16. On Existence of Linear Functionals on Abelian Groups

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In their paper [3, p. 147], S. Mazur and W. Orlicz have proved a fundamental existence theorem on linear functional in a linear space. In this Note, we shall prove now a similar theorem on Abelian groups.

Theorem. *Let $p(x)$ be a real valued subadditive functional on an Abelian group G, and $x(t)$ a function from an abstract set A to G. Let $\xi(t)$ be a real valued function on A. Then there is a linear functional $f(x)$ satisfying*

1) $f(x)\leq p(x)$ *for all* $x\in G$,

2) $\xi(x)\leq f(x(t))$ *for all* $t\in A$

if and only if

$$\sum_{i=1}^{n} m_i\xi(t_i)\leq p\Big(\sum_{i=1}^{n} m_i x(t_i)\Big) \tag{1}$$

for any finite set $t_i\in A$ and non-negative integers m_i, where $i=1, 2, \cdots, n$ and $n=1, 2,\cdots$.

The "only if" part is evident. To prove the "if" part, we use the technique by V. Ptak. In the middle of the proof, we need the following Aumann theorem [1].

Aumann theorem. *Let G be an Abelian group with a real valued subadditive functional $p(x)$,* i. e. *$p(x+y)\leq p(x)+p(y)$ and $p(0)=0$. Let H be a subgroup on G and $f(x)$ a linear functional on H,* i. e. *$f(x+y)=f(x)+f(y)$ for $x, y\in H$. If $f(x)\leq p(x)$ for all $x\in H$, then there is a linear extension F of f such that $F(x)\leq p(x)$ for each $x\in G$.*

An elegant proof by G. Mokobdzki is given in a note by P. Krée in the Séminaire Choquet [2]. The present writer can not approach to the original paper [1].

Proof of Theorem. Consider an auxiliary subadditive functional defined by

$$\tilde{p}(x)=\inf_{\substack{t_1,\cdots,t_n\\ m_1,\cdots,m_n}}\Big[p\Big(x+\sum_{i=1}^{n} m_i x(t_i)\Big)-\sum_{i=1}^{n} m_i\xi(t_i)\Big]$$

where m_i $(i=1, 2,\cdots, n)$ are non-negative integers. By the condition (1), we have

$$\sum_{i=1}^{n} m_i\xi(t_i)\leq p\Big(\sum_{i=1}^{n} m_i x(t_i)\Big)\leq p\Big(x+\sum_{i=1}^{n} m_i x(t_i)\Big)+p(-x).$$

Hence $p(x)$ is well defined. On the other hand, we have $-p(-x)\leq\tilde{p}(x)\leq p(x)$, so $p(0)=0$.

Further, we have

$$\tilde{p}(x+y)\leq p\Big(x+y+\sum_{i=1}^{n} m_i x(t_i)+\sum_{i=1}^{n} m_i' x(t_i')\Big)-\sum_{i=1}^{n} m_i \xi(t_i)$$
$$-\sum_{i=1}^{n} m_i'\xi(t_i')\leq p\Big(x+\sum_{i=1}^{n} m_i x(t_i)\Big)-\sum_{i=1}^{n} m_i \xi(t_i)$$
$$+p\Big(y+\sum_{i=1}^{n} m_i x(t_i)\Big)-\sum_{i=1}^{n} m_i'\xi(t_i')$$

for any t_i, t_i' and m_i, m_i' $(i=1, 2,\cdots, n)$. Hence

$$\tilde{p}(x+y)\leq \tilde{p}(x)+\tilde{p}(y).$$

Therefore, by Aumann's theorem, there is a linear functional $f(x)\leq \tilde{p}(x)$ for all $x\in G$. From $\tilde{p}(x)\leq p(x)$, we have $f(x)\leq p(x)$ for all $x\in G$.

On the other hand,

$$f(-x(t))\leq \tilde{p}(-x(t))\leq p(-x(t)+x(t))-\xi(t)=-\xi(t).$$

Hence $-f(x(t))\leq -\xi(t)$, and so $\xi(t)\leq f(x(t))$, which completes the proof.

Corollary. *Under the same condition on theorem, there is a linear functional $f(x)$ satisfying*

1) $f(x)\leq p(x)$ *for all* $x\in G$,

2) $\xi(t)=f(x(t))$ *for all* $t\in A$

if and only if (1) *holds true for any integers* m_i.

Corollary follows from Theorem without difficulty [3, p. 151].

References

[1] G. Aumann: Über die Erweiterung von additiven monotonen Funktionen auf regulär geordneten Halbgruppen. Arch. der Math., **8**, 422-427 (1957).

[2] Séminaire Choquet, Initiation à l'analyse. Faculté des Sci. de Paris, 1962.

[3] S. Mazur et W. Orlicz: Sur les espaces métriques linéaires II. Studia Math., **13**, 137-179 (1953).