45. *The Role of Mollifiers in Wightman Functions*

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§ 1. **Introduction.** The normal representation of Canonical commutation relations can be treated at least in three different ways [1]. Weyl's formulation and Segal's formulation are one of them.

The unitary operators which is constructed by Weyl's formulation is used for the construction of Wightman functions. These functions are one example of Wightman functions related to the axiomatic relativistic quantum field theory [2].

For the reinvestigation of the meaning of the unitary operator constructed by Weyl's formulation the exact definition of multiplication of the operator valued distributions is needed. But its concept is not clear. Here we define the three kinds of multiplication and show the method to find the most natural one.

The meaning of multiplication of field functions has not been definite. In the present paper, the meaning of non local field functions which appeared in Wightman functions above will be made clear by using our definitions. Furthermore, by using our definition, the meaning of commutation relations becomes clear.

On the other hand, the orthogonality between the domain of the free Hamiltonian and that of the total Hamiltonian is already proved in the case of the neutral scalor field with fixed point source [3–5]. Nobody yet has constructed a linear compatible topology satisfying the Hausdorff's axioms (**A–D**) in which the statevectors in interaction field can be approximated by the sequence of the state vectors in free field [6–7].

In Wightman's method the domain of the free Hamiltonian and that of the total Hamiltonian can be constructed separately in the same way. This situation is not sufficient in clarifying the relation between these two domains.

If we try to clarify this point forcibly, then it can be seen that the lack of the sufficiently strong linear compatible topology is an essential obstacle [9]. Hence, only free fields have been taken up so far. The case of interacting field will be referred to briefly, in this paper.

Here we shall show the limitation for the usual Gelfand's construction by using Weyl's formulation and show the effects of mollifiers upon this construction.

§ 2. **The multiplication of operator valued distributions.** We use the word "generalized infinite sum" to indicate the integral or sum of operator valued function. The operators treated here are the linear operators which have the range and the domain in Von Neumann's direct product space. They may have no domains. In a special case, they are infinite linear sum of creation and annihilation operators in the momentum space.

Let $\boldsymbol{D}(\boldsymbol{R}^n)$ be the space of the real testing functions defined in $\boldsymbol{R}^n$. $\varphi(f)$ means the integral $\int f(\boldsymbol{x})\varphi(\boldsymbol{x})d\boldsymbol{x}$, where $\varphi(\boldsymbol{x})$ is an operator valued function and $f(\boldsymbol{x})$ is a testing function in $\boldsymbol{D}(\boldsymbol{R}^3)$. The integral $\int f(\boldsymbol{x})\varphi(\boldsymbol{x})d\boldsymbol{x}$ is the generalized infinite linear sum of $\varphi(\boldsymbol{x})$ with coefficient $f(\boldsymbol{x})$.

Theorem 1. *If $\varphi(\boldsymbol{x})$ is an operator valued function and f is smooth, then $\varphi(f)$ is an operator with domain.*

For the proof of this Theorem, we need the exact definition of infinite linear sum i.e. $\varphi(f)$. But this argument is not the aim of this paper, and the proof will not give here.

Now consider the two operator valued functions defined in R^3 $\varphi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ which satisfy the following commutation relations;

$$[\varphi(\boldsymbol{x}), \varphi(\boldsymbol{x}')]=[\pi(\boldsymbol{x}), \pi(\boldsymbol{x}')]=0 \qquad (1)$$

$$[\varphi(\boldsymbol{x}), \pi(\boldsymbol{x}')]=i\delta(\boldsymbol{x}-\boldsymbol{x}'). \qquad (2)$$

Definition 1. *Let denote $A\cdot B$ the product of two operators A and B.* In our case A and B may have not domains.

Let's define the following three multiplications.

Definition 2. (1) Let's $\varphi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ be two given operator valued functions, and $(\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}}))$ and $(\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}'}))$ be the spaces of the testing functions of two operator valued functions $\varphi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$.

We then take $(\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}}\times\boldsymbol{R}^3_{\boldsymbol{x}'}))$ as the space of the testing functions of $\varphi(\boldsymbol{x})\times\psi(\boldsymbol{x}')$ whose meaning will be given below.

Let $\boldsymbol{L}(\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}}), \boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}'}))$ denote the subspace of $(\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}}\times\boldsymbol{R}^3_{\boldsymbol{x}'}))$ consisting of elements of the form $\Sigma_{i=1}^N C_i f_i(\boldsymbol{x})g_i(\boldsymbol{x}')$. Here N is a positive integer, C's are real constants, $f_i(\boldsymbol{x})$'s are contained in $(\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}}))$ for all i and $g_i(\boldsymbol{x}')$'s are in $(\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}'}))$ for all i.

If the element $h(\boldsymbol{x},\boldsymbol{x}')$ is contained in $\boldsymbol{L}(\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}}), \boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}'}))$, then the following operator valued functional is defined;

$$\langle\varphi(\boldsymbol{x})\times\psi(\boldsymbol{x}'), h(\boldsymbol{x},\boldsymbol{x}')\rangle=\Sigma_{i=1}^N C_i\langle\varphi(\boldsymbol{x})\times\psi(\boldsymbol{x}'), f_i(\boldsymbol{x})\times g_i(\boldsymbol{x}')\rangle$$
$$=\Sigma_{i=1}^N C_i\varphi(f)\cdot\psi(g).$$

If the element $h(\boldsymbol{x},\boldsymbol{x}')$ of $(\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}}\times\boldsymbol{R}^3_{\boldsymbol{x}'}))$ is not contained in $\boldsymbol{L}((\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}})), (\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}'})))$, then we can take a set of sequence $[\{h_n(\boldsymbol{x},\boldsymbol{x}')\}]$ in $\boldsymbol{L}((\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}})), (\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}'})))$, such that $\lim_{n\to\infty} h_n(\boldsymbol{x},\boldsymbol{x}')=h(\boldsymbol{x},\boldsymbol{x}')$ in $(\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}}\times\boldsymbol{R}^3_{\boldsymbol{x}'}))$ and define the set of the sequence $[\{\langle\varphi(\boldsymbol{x})\times\psi(\boldsymbol{x}'), h_n(\boldsymbol{x},\boldsymbol{x}')\rangle\}]$. If a sequence $\{\langle\varphi(\boldsymbol{x})\times\psi(\boldsymbol{x}'), h_n(\boldsymbol{x},\boldsymbol{x}')\rangle\}$ contained in the set $[\{\langle\varphi(\boldsymbol{x})\times\psi(\boldsymbol{x}'), h_n(\boldsymbol{x},\boldsymbol{x}')\rangle\}]$ converges to an operator Φ in a certain topology, then we *replace this sequence by Φ under the assumptions that only this topology is used.*

Here we do not refer to the uniqueness of this product etc. The above functional or the set of the functional sequences define the product operator valued function $\varphi(\boldsymbol{x})\times\psi(\boldsymbol{x}')$.

(2) *We denote by $\varphi(\boldsymbol{x})\circ\psi(\boldsymbol{x}')$ the following operator valued functional defined in a testing function's space $\boldsymbol{D}(\boldsymbol{R}^3)$.*

Namely, $\varphi(\boldsymbol{x})\circ\psi(\boldsymbol{x})\cdot(f(\boldsymbol{x}))=\langle\varphi(\boldsymbol{x})\times\psi(\boldsymbol{x}'), f(\boldsymbol{x})\times f(\boldsymbol{x}')\rangle=\varphi(f)\cdot\psi(f)$, where $f(\boldsymbol{x})$ belongs to $\boldsymbol{D}(\boldsymbol{R}^3)$.

(3) *We denote by $\varphi(\boldsymbol{x})\circledcirc\psi(\boldsymbol{x})$ the set of the following sequences of operator valued functionals defined in a testing function's space $\boldsymbol{D}(\boldsymbol{R}^3)$.*

Namely, $\varphi(\boldsymbol{x})\circledcirc\psi(\boldsymbol{x})\cdot(f(\boldsymbol{x}))=[\{\langle\varphi(\boldsymbol{x})*\rho_\varepsilon(\boldsymbol{x})\cdot\psi(\boldsymbol{x})*\rho_{\varepsilon'}(\boldsymbol{x}), f(\boldsymbol{x})\rangle\}]$ for $f(\boldsymbol{x})\in\boldsymbol{D}(\boldsymbol{R}^3)$, where $\rho_\varepsilon(\boldsymbol{x})$, $\rho_{\varepsilon'}(\boldsymbol{x})\in(\boldsymbol{D}(\boldsymbol{R}^3))$, $\lim_{\varepsilon\to 0}\rho_\varepsilon(\boldsymbol{x})=\delta$ and $\lim_{\varepsilon'\to 0}\rho_\varepsilon(\boldsymbol{x})=\delta$ in (D'), and $\lim_{\varepsilon\to 0}|\boldsymbol{x}|^m\rho_\varepsilon(\boldsymbol{x})=0$ and $\lim_{\varepsilon'\to 0}|\boldsymbol{x}|^m\rho_{\varepsilon'}(\boldsymbol{x})=0$ uniformly for an arbitrary fixed integer m [9].

Using these definitions the following problems arise under the basic rule such that we should use the multiplication which is the most faithful to the original one. The original definition of multiplication is not distribution-wise [8] [10].

1) What sort of multiplication should be used in the commutation relations (1) and (2)?

2) What sort of multiplication should be used in the definition of the unitary operators $\exp[i\varphi(\boldsymbol{x})]$ and $\exp[i\pi(\boldsymbol{x})]$?

Furthermore, the following problem arises.

3) Are the three multiplications defined above different with each other or not?

Since the product of two field functions used in the commutation relations are defined in $\boldsymbol{R}^3_{\boldsymbol{x}}\times\boldsymbol{R}^3_{\boldsymbol{x}'}$, it is very natural to use the testing functions defined in $\boldsymbol{R}^3_{\boldsymbol{x}}\times\boldsymbol{R}^3_{\boldsymbol{x}'}$. Hence only the multiplication (1) must be used in the commutation relations.

Namely, the commutation relation can be written as follows:

$$\langle[\varphi(\boldsymbol{x}),\varphi(\boldsymbol{x}')], h(\boldsymbol{x},\boldsymbol{x}')\rangle=\lim_{m\to\infty}\langle[\varphi(\boldsymbol{x}),\varphi(\boldsymbol{x}')], \Sigma_{n=1}^{N_m}C_{nm}f_{nm}(\boldsymbol{x})g_{nm}(\boldsymbol{x}')\rangle=0$$

$$\langle[\pi(\boldsymbol{x}),\pi(\boldsymbol{x}')], h(\boldsymbol{x},\boldsymbol{x}')\rangle=\lim_{m\to\infty}\langle[\pi(\boldsymbol{x}),\pi(\boldsymbol{x}')], \Sigma_{n=1}^{N_m}C_{nm}f_{nm}(\boldsymbol{x})g_{nm}(\boldsymbol{x}')\rangle=0$$

and

$$\langle[\varphi(\boldsymbol{x}),\pi(\boldsymbol{x}')], h(\boldsymbol{x},\boldsymbol{x}')\rangle=\lim_{m\to\infty}\langle i\delta(\boldsymbol{x}-\boldsymbol{x}'), \Sigma_{n=1}^{N_m}C_{nm}f_{nm}(\boldsymbol{x})g_{nm}(\boldsymbol{x}')\rangle$$

$$=i\int h(\boldsymbol{x},\boldsymbol{x})d\boldsymbol{x}.$$

Since $\exp(i\varphi(\boldsymbol{x}))$ and $\exp(i\pi(\boldsymbol{x}))$ are unitary operators for fixed $\boldsymbol{x}$ belonging to $\boldsymbol{R}^3_{\boldsymbol{x}}$, they are field functions on $\boldsymbol{R}^3_{\boldsymbol{x}}$. Hence it is very natural to use the testing functions on $\boldsymbol{R}^3_{\boldsymbol{x}}$. If it is allowed to take $(\boldsymbol{D}(\boldsymbol{R}^3_{\boldsymbol{x}}))$ as the testing function's space, then $\exp(i\varphi(\boldsymbol{x}))\cdot f(\boldsymbol{x})$ must be defined by the following form:

$$\exp(i\varphi(x))\cdot f(x)=[1+i\varphi(x)+(1/2!)i\varphi(x)\cdot i\varphi(x)+\cdots]\cdot f(x).$$

It is evident that the multiplications appeared in this series are not of type (1) in Definition 2. Taking into account that $\varphi(x)$ is operator valued, we shall show that these multiplications are not of type (2) in Definition 2.

Each term of the above series is not of the same form as in 2) of Definition 2. In fact we shall show that these two types of multiplications are essentially different with each other, i.e.

$$\langle\varphi(x)\cdot\psi(x), f(x)\rangle\neq\varphi(f)\cdot\psi(f).$$

Case 1. $T=\int\lambda dE(\lambda)$, where $dE(\lambda)$ is a spectral measure. Let's denote $\mathfrak{A}_T$ the set of operators $f(T)=\int f(\lambda)dE(\lambda)$ $f(\lambda)$ in C^∞. Suppose that the operator valued functions $\varphi(x)$ and $\psi(x)$ are in $\mathfrak{A}_T$ for each x. Then $\varphi(x)$ and $\psi(x)$ have the integral representations $\varphi(x)=\int q(x,\lambda)dE(\lambda)$ and $\psi(x)=\int p(x,\lambda)dE(\lambda)$, where $p(x,\lambda)$ and $q(x,\lambda)$ is in C^∞ for each x.

Then we have $\langle\varphi(x)\cdot\psi(x), f(x)\rangle=\langle\int q(x,\lambda)dE(\lambda)\cdot\int p(x,\lambda)dE(\lambda), f(x)\rangle$

$$=\int\int q(x,\lambda)p(x,\lambda)f(x)dxdE(\lambda).$$

The right hand side has a meaning different with that of

$$\int\left\{\int q(x,\lambda)f(x)dx\cdot\int p(x,\lambda)f(x)dx\right\}dE(\lambda).$$

For example, $(\delta\cdot\delta)f$ is not determined uniquely in general but $(\delta\cdot f)\cdot(\delta\cdot f)=f(0)^2$ has a definite meaning.

Case 2. Suppose that the field function $\varphi(x)$ can be decomposed in the formulas $\varphi(x)=\int\lambda dE_\varphi(x,\lambda)$ where $dE_\varphi(x,\lambda)$ is the spectral measure for each x. Then $\langle\varphi(x)\cdot\varphi(x), f(x)\rangle=\int\lambda^2 d\int E_\varphi(x,\lambda)f(x)dx$.

The right hand side is not equal to

$$\left\{\int\lambda d\int E_\varphi(x,\lambda)f(x)dx\right\}\cdot\left\{\int\lambda d\int E_\varphi(x,\lambda)f(x)dx\right\}.$$

That is, if $d\int E_\varphi(x,\lambda)f(x)dx$ is a spectral measure, the latter product of integrals will be reduced to a simple integral and will be equal to the former right hand side. But we can easily construct an example such that $d\int E_\varphi(x,\lambda)f(x)dx$ is not a spectral measure. For example, if $f(x)$ contained in $(D(R_x^3))$, satisfies the condition $\int f(x)dx\neq 1$ and if $E_\varphi(x,\lambda)=E(\lambda)$ is independent of x, then we obtain $\int d\int E_\varphi(x,\lambda)f(x)dx\neq 1$.

Hence it seems that the most faithful distribution-wise extension of $\exp(i\varphi(x))$ can be constructed only by using the multiplication of type (3) in Definition 2.

In Gelfand's construction by Weyl's formulation the multiplication of (2) is used. And the unitary operators used in this construction can be expressed as follows:

$$\exp(i\varphi(f)) = [1 + i\varphi(f) + (1/2!)i\varphi(f)\cdot i\varphi(f) + \cdots]$$
$$= [1 + i\{\varphi(\boldsymbol{x}) * f(\boldsymbol{x})\} + (1/2!)(i\{\varphi(\boldsymbol{x}) * f(\boldsymbol{x})\})^2 + \cdots]_{\boldsymbol{x}=0}.$$

Hence it is seen that the non-local field operator appears in this construction, and the multiplication used in Wightman's theory is not unified with that of commutation relations.

§ 3. Gelfand's construction

a) Let's consider linear space R with the Hausdorff topology τ.

Suppose that τ satisfies the conditions: (1) for any neighbourhood $U(x+y)$, there exist neighbourhoods $U_1(x)$ and $U_2(y)$ such that $U(x+y)$ contains the sum of these neighbourhoods $U_1(x)+U_2(y)$, (2) for any neighbourhood $U(\alpha x)$, there exists a neighbourhood $U_1(x)$ such that $U(\alpha x)$ contains the set $\alpha U_1(x)$. Then, we say that τ is a linear compatible topology in $\boldsymbol{R}$.

b) Let denote $(\boldsymbol{Z})$ the space consisting of the Fourier transform of elements in $(\boldsymbol{D})$, and let denote $\Psi(0(\boldsymbol{k}))$ the element in Von Neumann's direct product space corresponding to the free vacuum state [1].

Von Neumann's direct product space can be decomposed in incomplete direct product space: $\prod_{i\in I}\otimes\mathfrak{H}_i = \mathfrak{H}^0\oplus\mathfrak{H}^1\oplus\cdots$.

These $\mathfrak{H}^i$ satisfies the following properties:

(i) $\mathfrak{H}^i \perp \mathfrak{H}^j$ for $i \neq j$,

(ii) $\mathfrak{H}^0$ contains the free vacuum state $\psi(0(\boldsymbol{k}))$.

Theorem 2. *If $f\in(\boldsymbol{Z})$, then* $\exp(i\varphi(f))\psi(0(\boldsymbol{k}))\in\mathfrak{H}^0$.

Proof. The multiplications used in the expansion of this unitary operator is of type (2). And the exact form of the field function in practical use is expressed by the formula

$$\varphi(\boldsymbol{x}) = (1/\sqrt{(2\pi)^3})\left\{\int a^+(\boldsymbol{k})e^{i\boldsymbol{k}\boldsymbol{x}}d\boldsymbol{k} + \int a(\boldsymbol{k})e^{-i\boldsymbol{k}\boldsymbol{x}}d\boldsymbol{k}\right\}.$$

In $\varphi(\boldsymbol{x})*f(\boldsymbol{x}) = \mathfrak{F}^{-1}(\mathfrak{F}\varphi(\boldsymbol{x})\cdot\mathfrak{F}f(\boldsymbol{x}))$ the effects of sufficiently large $\boldsymbol{k}$ does not appear. Because $\mathfrak{F}f$ is contained in the space $(\boldsymbol{D}(\boldsymbol{k}^3))$.

$\exp(i\varphi(f))\psi(0(\boldsymbol{k}))$ is contained in the Hilbert space $\mathfrak{H}^0$ whose bases are the states obtained by operating the creation and annihilation operators to $\psi(0(\boldsymbol{k}))$ for finite $\boldsymbol{k}$'s whose absolute values are bounded.

From this theorem we cannot construct $\mathfrak{H}^i$ (for $i\neq 0$) from a state by using the above procedure.

(**c**) In Von Neumann's direct product space, the orthogonal property is not usual. That is, $\langle \prod_{n\in N}\otimes\varphi_n, \prod_{n\in N}\otimes\psi_n\rangle = \prod_{n\in N}\langle\varphi_n,\psi_n\rangle = 0$ for

$\prod_{n\in N}\otimes\varphi_n$, $\prod_{n\in N}\otimes\psi_n$ such that $\langle\varphi_n, \psi_n\rangle \neq 0$ (for all n).

We cannot, however, construct a linear compatible topology in which the element in $\mathfrak{H}^i$ is approximated by sequences of the elements in $\mathfrak{H}^0$. In the proof of the uniqueness of the vacuum state, it is seen that the linear compatibility is essentially used.

Since the linear compatibility seems to be valied in each incomplete direct product space, the uniqueness theorem can be applied to only the free fields. The use of linear compatibility is shown by the following example: "from $(1/N)\ \Sigma_{i=1}^N T_{\vec{a}_i}\psi_0' = \psi_0'$ we can deduce $(1/N)\ \Sigma_{i=1}^N T_{\vec{a}_i}\psi_\varepsilon \sim \psi_\varepsilon$, where $||\psi_0'||^2 = 1$, $||\psi_\varepsilon||^2 \sim 1$, $||\psi_\varepsilon - \psi_0'|| < \varepsilon$, $\psi_\varepsilon = \Sigma_1^n C_i U(f_i)\psi_0$ and $T_{\vec{a}_i}$ is the translation of the length $\vec{a}_i$" [2]. (The symbol $\sim$ is taken to mean equal to within terms of infinitesimal order ε.)

§ 4. The extension of the space of mollifiers. If we wish to adopt the above theorem to the interaction field, we must use the space $\boldsymbol{D}$ which contains the generalized function δ instead of $(\boldsymbol{Z})$, as the space of mollifiers.

In this case, however, we meet the difficulty of the multiplication of singular field functions. But we will be able to avoid it by using the result in [9].

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