44. On the Theorems of Constantinescu-Cornea

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1. Let f be a non-constant analytic mapping from a hyperbolic Riemann surface R into an arbitrary Riemann surface R'. C. Constantinescu and A. Cornea defined[1] a cluster set and developed the theorem of Riesz and the theorem of Fatou. Their cluster set is defined by means of the operator I and the argument is carried out mechanically. We shall give here an intuitive interpretation of this cluster set by the notion of thinness due to L. Naïm.[2]

2. We can define the Martin boundary Δ of R, and the set of minimal boundary points Δ_1.[3] For $s \in \Delta_1$ and an open subset G in R Constantinescu-Cornea defined

$$\underset{G}{I}K_s = \sup\{u(p);\ u \in HP(\eta),\ u \leq K_s \text{ in } G\},$$

where K_s is the minimal positive harmonic function in R corresponding to s and η is an identity mapping from G into R. By definition, $u \in HP(\eta)$ if and only if for every relatively compact open set $G_1 \subset R$, $H_u^{G \cap G_1} = u$ in $G \cap G_1$ where $H_u^{G \cap G_1}$ denotes the solution of the Dirichlet problem with the boundary function u on $\partial(G \cap G_1) \cap G$[4] and 0 else where. Further, if $\underset{G}{I}K_s \neq 0$ they set $s \in \Delta_1(G)$ and the cluster set is defined as follows:

$$\hat{M}_f(s) = \bigcap_{s \in \Delta_1(G)} \overline{f(G)},$$

$\overline{f(G)}$ is the closure of $f(G)$ in $\hat{R}'$ (compactification of R').[5]

We shall here remark that the set $\Delta_1(G)$ permits the potential theoretic view. In fact, Constantinescu-Cornea showed that the following equality holds in G:

$$\underset{G}{I}K_s = K_s - H_{K_s}^G.$$[6]

1) C. Constantinescu and A. Cornea: Über das Verhalten der analytischen Abbildungen Riemannscher Flächen auf dem idealen Rand von Martin. Nagoya Math. J., **17**, 1-87 (1960).

2) L. Naïm: Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel. Ann. Inst. Fourier, **7**, 183-281 (1957).

3) For the construction and the properties of the Martin boundary see L. Naïm, l.c., also M. Parreau: Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann. Ann. Inst. Fourier, **3**, 103-197 (1952). R. S. Martin: Minimal positive harmonic functions. Trans. Amer. Math. Soc., **49**, 137-172 (1941).

4) $\partial(G \cap G_1)$ denotes the boundary of $G \cap G_1$.

5) Cf. Constantinescu-Cornea, l.c., S. 44.

6) Cf. Constantinescu-Cornea, l.c., Hilfssatz 4, S. 21.

On the other hand, L. Naïm defined the thinness of a set at the minimal boundary point and gave the criterion:[7] "the set $F\subset R$ is thin at $s\in\Delta_1$ if and only if $\mathcal{E}_{K_s}^{R-F}\not\equiv K_s$, where $\mathcal{E}_{K_s}^{R-F}$ denotes the extremization of K_s on $R-F$".[8] If $R-F$ is open then $\mathcal{E}_{K_s}^{R-F}=H_{K_s}^{R-F}$ in $R-F$. Hence we can get the relation: *$s\in\Delta_1(G)$ implies that the set $R-G$ is thin at s and vice versa.*

From this point of view some properties of $\Delta_1(G)$ are easily seen, for example, $\Delta_1(G_1)\cap\Delta_1(G_2)=\Delta_1(G_1\cap G_2)$ is verified as follows: if F is thin at s and $F_2\subset F_1$ then F_2 is also thin at s, therefore

$$\Delta_1(G_1)\cap\Delta_1(G_2)\supset\Delta_1(G_1\cap G_2).$$

On the other hand, if F_1 and F_2 are thin at s then $F_1\cup F_2$ is thin at s, therefore

$$\Delta_1(G_1)\cap\Delta_1(G_2)\subset\Delta_1(G_1\cap G_2).$$

3. Next, we shall give a proof of the theorem of Riesz from our point of view.

Lemma. *If u is a positive harmonic function in R and*

$$u(p)=\int_{\Delta_1}K_s(p)d\mu(s)$$

and F is a set of $\mathcal{F}_\sigma$[9] in R then

$$\mathcal{E}_u^F(p)=\int_{\Delta_1}\mathcal{E}_{K_s}^F(p)\,d\mu(s).$$

Proof. It is known that $\mathcal{E}_u^F(p)=\int u(q)d\epsilon_p'(q)$ where $d\epsilon_p'$ denotes the mass-distribution defined by sweeping out the unit mass at p on F. For $(q,s)\in F\times\Delta_1$, $K_s(q)$ is a positive measurable function in (q,s) we can adapt the Fubini's theorem:

$$\begin{aligned}\mathcal{E}_u^F(p)=\int u(q)d\epsilon_p'(q)&=\int\Big[\int_{\Delta_1}K_s(q)d\mu(s)\Big]d\epsilon_p'(q)\\&=\int_{\Delta_1}\Big[\int K_s(q)d\epsilon_p'(q)\Big]d\mu(s)\\&=\int_{\Delta_1}\mathcal{E}_{K_s}^F(p)\,d\mu(s).\end{aligned}$$

Before stating the theorem of Riesz, we shall define some notions. *Let us assume that R' is also hyperbolic,* then we can construct the Martin space $\hat{R}'$.[10] The set $\hat{A}'$ in $\hat{R}'$ is *polar* if there exists a positive superharmonic function S' such that

$$\lim_{p'\to\hat{q}'}S'(p')=+\infty \text{ holds for each } \hat{q}'\in\hat{A}'.$$

The *canonical* representation of 1:

$$1\equiv\int_{\Delta_1}K_s dx(s)$$

7) Cf. L. Naïm, l.c., p. 201 and théorème 5, p. 205.
8) Cf. L. Naïm, l.c., p. 192.
9) The set of $\mathcal{F}_\sigma$ is defined as the set which is a union of countable closed sets.
10) Cf. L. Naïm, l.c., p. 192.

gives the mass-distribution χ.[11] Let A be the Borel subset of Δ, if $\chi(A)=0$ then we shall say that A is *of harmonic measure zero.*

Theorem of Riesz. *Let $\widehat{A}'$ be a polar set on $\widehat{R}'$. For some set $A\subset\Delta_1$, if the relation $\widehat{M}_f(s)\subset\widehat{A}'$ holds for every $s\in A$, then A is of harmonic measure zero.*

Proof. Let S' be a positive superharmonic function on R' such that

$$S'(f(p_0))\neq+\infty^{12)}$$

and

$$\lim_{p'\to\hat{q}'\in\hat{A}'} S'(p')=+\infty.$$

We write for each $\alpha>0$

$$G'_\alpha=\{p'\in R';\ S'(p')>\alpha\}$$
$$G_\alpha=f^{-1}(G'_\alpha).$$

Let s belong to A then for every $\alpha>0$ there exists an $\varepsilon>0$ such that $G'(\widehat{M}_f(s),\varepsilon)\subset G'_\alpha$, where $G'(\widehat{M}_f(s),\varepsilon)$ denotes the intersection of R' and the ε-neighbourhood of $\widehat{M}_f(s)$ in the Martin space $\widehat{R}'$, then $s\in\Delta_1(f^{-1}(G'(\widehat{M}_f,\varepsilon)))\subset\Delta_1(G_\alpha)$ therefore we shall have for every $\alpha>0$, $A\subset\Delta_1(G_\alpha)$. If we show that $\lim_{\alpha\to+\infty}\chi(\Delta_1(G_\alpha))=0$, then we shall get A is of harmonic measure zero. The proof of $\lim_{\alpha\to+\infty}\chi(\Delta_1(G_\alpha))=0$ is as follows: we shall write $S(p)=S'(f(p))$, then S is a positive superharmonic function on R, and

$$\frac{1}{\alpha}S>1 \qquad \text{on } G_\alpha$$

therefore

$$\frac{1}{\alpha}S\geq\mathcal{E}_1^{R-G_\alpha} \qquad \text{in } R.$$

From $1=\int_{\Delta_1}K_s d\chi(s)$ and the preceding lemma

$$\mathcal{E}_1^{R-G_\alpha}=\int_{\Delta_1}\mathcal{E}_{K_s}^{R-G_\alpha}d\chi(s)=\int_{\Delta_1(G_\alpha)}\mathcal{E}_{K_s}^{R-G_\alpha}d\chi(s)+\int_{\Delta_1-\Delta_1(G_\alpha)}\mathcal{E}_{K_s}^{R-G_\alpha}d\chi(s).$$

For $s\in\Delta_1(G)$, $R-G_\alpha$ is thin at s and R is not thin at s therefore G_α is not thin at s hence $\mathcal{E}_{K_s}^{R-G_\alpha}\equiv K_s$. Hence

$$\frac{1}{\alpha}S(p)\geq\mathcal{E}_1^{R-G_\alpha}(p)\geq\int_{\Delta_1(G_\alpha)}K_s(p)d\chi(s).$$

For $p=p_0$ we get $\frac{1}{\alpha}S(p_0)\geq\chi(\Delta_1(G_\alpha))$ and $\alpha\to+\infty$ we get the desired result.

4. Constantinescu-Cornea showed that for an open set $G\subset R$,

11) Cf. L. Naïm, l.c., p. 193.

12) p_0 is the point of normalization for Martin's kernel $K_s(p)$ — that is for p, q in R and for Green function of R, $g(p,q)$, $K_q(p)=\dfrac{g(p,q)}{g(p_0,q)}$.

and for a positive harmonic function $u(p)=\int_{\Delta_1} K_s(p)d\mu(s)$,

$$\underset{G\,G}{EI}u=\int_{\Delta_1(G)} K_s d\mu(s).^{13)}$$

As in the preceding proof we can see

$$\underset{G\,G}{EI}u\leq \mathcal{E}_u^{R-G}\leq u.$$

Theorem 1. *Let $R\in U$[14] and G be an open set. Then if one of the components of G belongs to the class U then*

$$\mathcal{E}_1^G \mathrel{\not\equiv} 1,$$

and if $\Delta-\Delta_1(G)$ is of harmonic measure zero then at least one components of G belongs to U.[15]

Proof. Let R_0 be a component of G such that $R_0\in U$ and u_0 be a bounded minimal positive harmonic function in R_0, then we can form

$u=\inf\{v;\ v$ is a non-negative superharmonic function in R and $v\geq u_0$ in $R_0\}$

$\hat{u}$, the regularization of u, is bounded minimal in R. Therefore $\hat{u}=\lambda K_{s_0}$, $s_0\in\Delta_1$, but in this case we can show that $s_0\in\Delta_1(G)$. Now we shall assume that $\mathcal{E}_1^G\equiv 1$. This implies that $\chi(\{s;\ s\in\Delta_1,\ K_s\mathrel{\not\equiv}\mathcal{E}_{K_s}^G\})=0$ and this leads to a contradiction since $K_{s_0}\mathrel{\not\equiv}\mathcal{E}_{K_{s_0}}^G$ and $\chi(\{s_0\})>0$. Next, if $\chi(\Delta-\Delta_1(G))=0$ then $\int_{\Delta_1(G)} K_s d\chi(s)\equiv 1$. Let K_{s_0} be a bounded minimal positive harmonic function in R, then $\chi(\{s_0\})>0$, therefore $s_0\in\Delta_1(G)$ and

$$u_0=K_{s_0}-\mathcal{E}_{Ks_0}^G$$

is a bounded minimal positive harmonic function in some component of G.

Remark. For an open set G we have

$$\Delta_1=\Delta_1(G)\cup\Delta_1(R-\bar{G})\cup A$$

where $A=\Delta_1-[\Delta_1(G)\cup\Delta_1(R-\bar{G})]$ and these three sets are mutually disjoint. The necessary condition of the above theorem means that $\chi(\Delta_1(G))\neq 0$ and the sufficient condition means that $\chi(\Delta_1(G))=1$. More precise condition is required. We shall remark that if $\chi(A)=0$, for instance if the relative boundary of G is compact, the two conditions $\underset{G\,G}{EI}1\equiv 1$ and $\mathcal{E}_1^{R-G}\equiv 1$ are equivalent and we shall get:

Theorem 2. *Let $R\in U$ and G be an open set of which relative boundary is compact. If $\mathcal{E}_1^{R-G}\equiv 1$ then at least one of the components of G belongs to U.*

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13) Cf. Constantinescu-Cornea, l.c., Satz 15′, S. 42.

14) A Riemann surface $R\notin O_G$ belongs to the class U if R has at least one bounded minimal positive harmonic function.

15) The hypothesis of the latter part means that $\underset{GG}{EI}1\equiv 1$. Cf. Constantinescu-Cornea, l.c., Folgesatz 7, S. 70.