43. On the Lebesgue Constants for Quasi-Hausdorff Methods of Summability. II

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§ 5. For the proof of Theorem 1, we shall prove the following

Lemma.

$$(5.1)\qquad L_\delta^*(n;\psi)=\frac{2}{\pi}\int_1^{\sqrt{n}}\frac{du}{u}\left|\int_\delta^1 \sin\frac{u}{r}\,d\psi(r)\right|+ \\ +\frac{2}{\pi^2}\,|\psi(1)-\psi(1-0)|\log n+o\,(\log n).$$

It may be noted that the upper limits of the Stieltjes integrals in (3.4) and (5.1) are different.

Proof. We shall use the method of L. Lorch and D. J. Newman [5]. In order to simplify the following calculations, we shall prove

$$(5.2)\qquad L_\delta^*(n-1;\psi)=\frac{2}{\pi}\int_1^{\sqrt{n}}\frac{du}{u}\left|\int_\delta^1 \sin\frac{u}{r}\,d\psi(r)\right|+ \\ +\frac{2}{\pi^2}\,|\psi(1)-\psi(1-0)|\log n+o\,(\log n).$$

It is easily seen that (5.1) and (5.2) are equivalent for large n.

Replacing the factor $\{\sin(2n+1)u\}/\sin u$ by $\{\sin 2(n+1)u\}/u$ in (3.4) induces a bounded error, we obtain, from (2.2),

$$L_\delta^*(n-1;\psi)=\frac{2}{\pi}\int_0^{\pi/2}|K_n(u)|\frac{du}{u}+O(1),$$

where

$$(5.3)\qquad K_n(u)=\int_\delta^1\left(\frac{1}{1+\frac{4(1-r)}{r^2}\sin^2 u}\right)^{\frac{n}{2}}\sin\frac{2nu}{r}\,d\psi(r).$$

For fixed ε and A with $0<\varepsilon<1<A$, we put

$$\int_0^{\pi/2}|K_n(u)|\frac{du}{u}=\int_0^{\frac{\varepsilon}{\sqrt{n}}\delta^*}+\int_{\frac{\varepsilon}{\sqrt{n}}\delta^*}^{\frac{A}{\sqrt{n}}\delta^*}+\int_{\frac{A}{\sqrt{n}}\delta^*}^{\pi/2}=I_1+I_2+I_3,$$

where $\delta^*=\delta/\sqrt{2(1-\delta)}$.

As to I_1: In the interval $0\le u\le\frac{\varepsilon}{\sqrt{n}}\delta^*$, we have

$$1\ge\left(\frac{1}{1+\frac{4(1-r)}{r^2}\sin^2 u}\right)^{\frac{n}{2}}\ge 1-\varepsilon^2,$$

whence

$$\left|\,|K_n(u)|-\left|\int_\delta^1 \sin\frac{2nu}{r}\,d\psi(r)\right|\,\right|\leq \varepsilon^2 V(\psi),$$

where $V(\psi)$ is the total variation of $\psi(r)$ in the interval $0\leq r\leq 1$. Obviously, for $0\leq u\leq \frac{\pi}{2}$,

$$\left|\,|K_n(u)|-\left|\int_\delta^1 \sin\frac{2nu}{r}\,d\psi(r)\right|\,\right|\leq \frac{2nu}{\delta} V(\psi).$$

Hence,

$$\begin{aligned} I_1&=\int_0^{\frac{\varepsilon}{\sqrt{n}}\delta^*} |K_n(u)|\frac{du}{u} \\ &=\int_0^{\frac{\varepsilon}{\sqrt{n}}\delta^*} \frac{du}{u}\left|\int_\delta^1 \sin\frac{2nu}{r}\,d\psi(r)\right|+E_0, \end{aligned}$$

where

$$\begin{aligned} |E_0|&\leq V(\psi)\int_0^{\frac{\varepsilon}{n}\delta^*}\frac{2nu}{\delta}\frac{du}{u}+V(\psi)\int_{\frac{\varepsilon}{n}\delta^*}^{\frac{\varepsilon}{\sqrt{n}}\delta^*}\varepsilon^2\frac{du}{u} \\ &=\left(\frac{2\varepsilon\delta^*}{\delta}+\frac{\varepsilon^2}{2}\log n\right)V(\psi). \end{aligned}$$

Next,

$$\int_{\frac{\varepsilon}{\sqrt{n}}\delta^*}^{\frac{1}{2\sqrt{n}}}\frac{du}{u}\left|\int_\delta^1 \sin\frac{2nu}{r}\,d\psi(r)\right|\leq\left(\log\frac{1}{2\varepsilon\delta^*}\right)V(\psi),$$

so that, replacing $2nu$ by u,

$$I_1=\int_1^{\sqrt{n}}\frac{du}{u}\left|\int_\delta^1 \sin\frac{u}{r}\,d\psi(r)\right|+E_1, \tag{5.4}$$

where

$$\begin{aligned} |E_1|&\leq |E_0|+\left(\log\frac{1}{2\varepsilon\delta^*}\right)V(\psi)+\frac{1}{\delta}V(\psi) \\ &\leq\left\{\frac{2\varepsilon\delta^*}{\delta}+\frac{\varepsilon^2}{2}\log n+\log\frac{1}{2\varepsilon\delta^*}+\frac{1}{\delta}\right\}V(\psi). \end{aligned}$$

As to I_2: Since $|K_n(u)|\leq V(\psi)$, we have

$$|I_2|\leq\left(\log\frac{A}{\varepsilon}\right)V(\psi). \tag{5.5}$$

As to I_3: From (5.3), we have

$$\begin{aligned} K_n(u)=&[\psi(1)-\psi(1-0)]\sin 2nu+ \\ &+\int_\delta^{1-0}\left(\frac{1}{1+\frac{4(1-r)}{r^2}\sin^2 u}\right)^{\frac{n}{2}}\sin\frac{2nu}{r}\,d\psi(r). \end{aligned}$$

Further, for $\frac{A\delta^*}{\sqrt{n}} \leq u \leq \frac{\pi}{2}$, we have

$$\left(\frac{1}{1+\frac{4(1-r)}{r^2}\sin^2 u}\right)^{\frac{n}{2}} \leq \left(\frac{1}{1+\frac{4(1-r)}{r^2}\left(\frac{2}{\pi}\right)^2\frac{(A\delta^*)^2}{n}}\right)^{\frac{n}{2}}$$

$$\leq \exp\left\{-\frac{4(1-r)}{r^2}\frac{(A\delta^*)^2}{\pi^2}\right\}$$

for large n.

Hence

$$\left|\int_\delta^{1-0}\left(\frac{1}{1+\frac{4(1-r)}{r^2}\sin^2 u}\right)^{\frac{n}{2}}\sin\frac{2nu}{r}\,d\psi(r)\right|$$

$$\leq \int_\delta^{1-0}\exp\left\{-\frac{4(1-r)}{r^2}\frac{(A\delta^*)^2}{\pi^2}\right\}|d\psi(r)| = \phi(A),$$

say. It may be noted that $\phi(A)$ is independent of n, and tends to zero as $A\to\infty$ from the Lebesgue principle of dominated convergence.

Hence, we obtain

$$I_3 = |\psi(1)-\psi(1-0)|\int_{\frac{A}{\sqrt{n}}\delta^*}^{\frac{\pi}{2}}\frac{|\sin 2nu|}{u}\,du + E_3,$$

where

$$|E_3| \leq \phi(A)\int_{\frac{A}{\sqrt{n}}\delta^*}^{\frac{n}{3}}\frac{du}{u} \leq \phi(A)\log n$$

for all large n. Here

$$\int_{\frac{A}{\sqrt{n}}\delta^*}^{\frac{\pi}{2}}\frac{|\sin 2nu|}{u}\,du = \int_{\frac{A}{\sqrt{n}}\delta^*}^{\frac{\pi}{2}}\frac{|\sin 2nu| - \frac{2}{\pi}}{u}\,du +$$

$$+\frac{2}{\pi}\log\frac{\pi\sqrt{n}}{2A} = \frac{1}{\pi}\log n + E_4,$$

where $|E_4| \leq \log A + C$, with

$$C = \sup_{V>U\geq 1}\left|\int_U^V \frac{|\sin v| - \frac{2}{\pi}}{v}\,dv\right| < \infty.$$

Thus,

$$I_3 = \frac{1}{\pi}|\psi(1)-\psi(1-0)|\log n + E_5, \tag{5.6}$$

where

$$|E_5| \leq |\psi(1)-\psi(1-0)|\{\log A + C\} + \phi(A)\log n.$$

Since

$$\frac{L_\delta^*(n-1;\psi)}{\log n}=\frac{2}{\pi}\frac{1}{\log n}\{I_1+I_2+I_3\}+O\left(\frac{1}{\log n}\right),$$

we obtain, from (5.4), (5.5), and (5.6),

$$\limsup_{n\to\infty}\left|\frac{2}{\pi\log n}\int_1^{\sqrt{n}}\frac{du}{u}\left|\int_\delta^1\sin\frac{u}{r}\,d\psi(r)\right|+\right.$$
$$\left.+\frac{2}{\pi^2}|\psi(1)-\psi(1-0)|-\frac{L_\delta^*(n-1;\psi)}{\log n}\right|$$
$$\leq\frac{\varepsilon^2}{\pi}V(\psi)+\frac{2}{\pi}\phi(A).$$

Here we make $\varepsilon\to 0$ and $A\to\infty$, and obtain our lemma.

Proof of Theorem 1. Since the proof is quite similar to that of Theorem 2, we shall sketch it briefly. Let

$$F(T)=\int_0^T du\left|\int_0^1\sin\frac{u}{r}\,d\psi(r)\right|,$$

then

$$F(T)=T\mathcal{M}\left\{\left|\sum_k[\psi(\xi_k+0)-\psi(\xi_k-0)]\sin\frac{u}{\xi_k}\right|\right\}+o(T)$$
$$=T\mathcal{M}(\psi)+o(T),\quad\text{say.}$$

Hence we obtain, by partial integration,

$$\int_1^{\sqrt{n}}\frac{du}{u}\left|\int_\delta^1\sin\frac{u}{r}\,d\psi(r)\right|=\int_1^{\sqrt{n}}\frac{F'(T)}{T}\,dT$$
$$=\frac{1}{2}\mathcal{M}(\psi)\log n+o(\log n),$$

where δ is an appropriate positive constant.

From the previous lemma, this completes the proof of Theorem 1.

References

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