37. On Completeness of Royden's Algebra

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(Comm. by Kinjirô KUNUGI, M.J.A., March 12, 1964)

Let R be a Riemann surface and $\boldsymbol{M}(R)$ be Royden's algebra associated with R, i.e. the totality of bounded continuous a.c.T. functions*) on R with finite Dirichlet integrals. We say that a sequence $\{\varphi_n\}$ of functions in $\boldsymbol{M}(R)$ converges to a function φ in C-topology if it converges uniformly on any compact subset of R. If a sequence $\{\varphi_n\}$ is bounded and converges to φ in C-topology, then we say that $\{\varphi_n\}$ converges to φ in B-topology. If the Dirichlet integral $\iint\limits_R d(\varphi_n-\varphi)\wedge {}^*\overline{d(\varphi_n-\varphi)}$ tends to zero, then we say that $\{\varphi_n\}$ converges to φ in D-topology. Finally a sequence $\{\varphi_n\}$ converges to φ in BD-topology, if it converges in B-topology and D-topology. Let $\boldsymbol{M}_0(R)$ be the totality of functions in $\boldsymbol{M}(R)$ with compact supports in R and $\boldsymbol{M}_\Delta(R)$ be the potential subalgebra of $\boldsymbol{M}(R)$, i.e. the closure of $\boldsymbol{M}_0(R)$ in BD-topology. Let $\Gamma(R)$ be the totality of differentials α of the first order on R with finite Dirichlet integrals. Then $\Gamma(R)$ is a Hilbert space with an inner product $(\alpha,\beta)=\iint\limits_R \alpha\wedge{}^*\bar\beta$. Clearly $\{df;\ f\in\boldsymbol{M}(R)\}\subset\Gamma(R)$. The algebras $\boldsymbol{M}(R)$ and $\boldsymbol{M}_\Delta(R)$ are complete with respect to BD-topology respectively. (cf. Lemma 1.5, p. 208 in Nakai [3]). Moreover we have the following theorem.

Theorem 1. *If* $\varphi_n\in\boldsymbol{M}(R)$ *and if* (1) $\varphi_n\to\varphi$ *in C-topology and* φ *is bounded,* (2) *the Dirichlet integral* $D_R(\varphi_n)$ *is bounded, then* (3) $\varphi\in\boldsymbol{M}(R)$, (4) $d\varphi_n\to d\varphi$ weakly in $\Gamma(R)$.

Proof. Generally, a bounded subset of a Hilbert space is weakly compact (cf. ch. 1, § 4 in Nagy [2]). Since $\{d\varphi_n\}$ is bounded in $\Gamma(R)$ by condition (2), there exists a subsequence $\{d\varphi_{n_k}\}$ such that $\{d\varphi_{n_k}\}$ converges to some $\alpha\in\Gamma(R)$ weakly in $\Gamma(R)$. We shall show that $\varphi\in\boldsymbol{M}(R)$ and $d\varphi=\alpha$. Let $z=x+iy$ be a local parameter in R and let G be a square domain: $-1<x<1$, $-1<y<1$ in the coordinate neighborhood of z. We put $\alpha=a(x,y)dx+b(x,y)dy$ in G and we take a differential β such that $\beta=\bar\phi dy$ in G and $\beta=0$ outside of G, where ϕ is in the class C^∞ and its support is contained in G. Then we have

$$(\alpha,\beta)=\iint\alpha\wedge{}^*\bar\beta=\iint\limits_G a\phi dxdy.$$

By integration by parts, we get

*) For the definition of a.c.T. functions, refer to A. Pfluger: Comment. Math. Helvt., **33**, 23-33 (1959).

$$(d\varphi_{n_k}, \beta)=\iint d\varphi_{n_k}\wedge {}^*\bar{\beta}=\iint_G \Big(\frac{\partial \varphi_{n_k}}{\partial x}\phi\Big)dxdy=-\iint_G \varphi_{n_k}\frac{\partial \phi}{\partial x}dxdy.$$

On the other hand

$$\lim_k\,(d\varphi_{n_k}, \beta)=(\alpha, \beta).$$

Therefore

$$\iint_G a\phi dxdy=(\alpha, \beta)=\lim_k\,(d\varphi_{n_k}, \beta)=-\lim_k\iint_G \varphi_{n_k}\frac{\partial \phi}{\partial x}dxdy.$$

Since $\{\varphi_{n_k}\}$ converges to φ uniformly in G, the last term of the above is equal to $-\iint_G \varphi\frac{\partial\phi}{\partial x}dxdy$. Hence

$$\iint_G a\phi dxdy=-\iint_G \varphi\frac{\partial\phi}{\partial x}dxdy.$$

The above equality holds for any function ϕ which is in the class C^∞ and its support is contained in G. Hence the partial derivative $\frac{\partial\varphi}{\partial x}$ of φ in the sense of the theory of distributions is equal to a measurable function $a(x, y)$. By Nikodym's theorem (cf. Theorem 5, p. 58 in Schwartz [8]), $\varphi(x, y)$ is absolutely continuous with respect to x in $-1<x<1$ for almost all values of fixed y in $-1<y<1$ and the partial derivative $\frac{\partial\varphi}{\partial x}$ in the usual sense is equal to $a(x, y)$ for almost all values of (x, y) in G. Similarly, $\varphi(x, y)$ is absolutely continuous with respect to y in $-1<y<1$ for almost all values of fixed x in $-1<x<1$ and $\frac{\partial\varphi}{\partial y}$ is equal to $b(x, y)$ for almost all values of (x, y) in G. Since $\varphi(x, y)$ is continuous and $a(x, y)$ and $b(x, y)$ are locally square integrable, $a(x, y)$ and $b(x, y)$ and so $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\varphi}{\partial y}$ are all locally integrable. Hence φ is an a.c.T. function. On the other hand, $d\varphi=\alpha\in\Gamma(R)$, i.e. $D_R(\varphi)<+\infty$. By condition (1), φ is bounded and continuous. Hence $\varphi\in\boldsymbol{M}(R)$. Next, any subsequence of $\{d\varphi_n\}$ contains a subsequence which converges weakly in $\Gamma(R)$. From the above proof, this subsequence converges to $d\varphi$ weakly in $\Gamma(R)$. Hence $\{d\varphi_n\}$ itself converges to $d\varphi$ weakly in $\Gamma(R)$.

Corollary 1. *$\boldsymbol{M}(R)$ is a normed ring with respect to the norm* $||f||=\sup_R|f|+\sqrt{D_R(f)}$ (Lemma 1.1, p. 203 in Nakai [3]).

Corollary 2. *$\boldsymbol{M}(R)$ is complete with respect to the BD-topology* (Lemma 1.5, p. 208 in Nakai [3]).

Theorem 2. *If $\varphi_n\in\boldsymbol{M}_\Delta(R)$ and if* (1) *$\varphi_n\to\varphi$ in C-topology and φ is bounded,* (2) *$D_R(\varphi_n)$ is bounded, then* (3) $\varphi\in\boldsymbol{M}_\Delta(R)$.

Proof. By Theorem 1, $\varphi\in\boldsymbol{M}(R)$. Let $\{R_m\}_{m=0}^\infty$ be a normal exhaustion of R such that $R_1-\bar{R}_0$ is an annulus. Let $w(p)$ be a continuous function on R such that $w(p)=0$ on $\bar{R}_0$, $w(p)$ is harmonic on $R_1-\bar{R}_0$ and $w(p)=1$ on $R-R_1$. Since

$$\varphi = w\varphi + (1-w)\varphi$$

and clearly

$$(1-w)\varphi \in \boldsymbol{M}_0(R) \subset \boldsymbol{M}_\Delta(R),$$

it is sufficient to prove that $w\varphi \in \boldsymbol{M}_\Delta(R)$. Let u_m be a continuous function such that $u_m = 0$ on $\bar{R}_0$ and u_m is harmonic on $R_m - \bar{R}_0$ and $u_m = \psi$ on $R - R_m$, where $\psi = w\varphi$. Then $u_m - u_{m+p}$ is equal to zero outside of $R_{m+p} - \bar{R}_0$ and u_{m+p} is harmonic in $R_{m+p} - \bar{R}_0$. By Green's formula

$$D_R(u_m - u_{m+p}, u_{m+p}) = \int_{\partial(R_{m+p} - R_0)} (u_m - u_{m+p})^* du_{m+p} = 0.$$

Hence

$$0 = D_R(u_m - u_{m+p}, u_{m+p}) = D_R(u_m, u_{m+p}) - D_R(u_{m+p}),$$

so we get

$$D_R(u_m - u_{m+p}) = D_R(u_m) - D_R(u_{m+p}).$$

Since

$$D_R(u_m) \geq D_R(u_{m+p}) \geq 0,$$

$\{D_R(u_m)\}$ converges and

$$D_R(u_m - u_{m+p}) \to 0 \quad \text{as } m \to \infty.$$

On the other hand $\{u_m\}$ is bounded and u_m is harmonic on $R_m - \bar{R}_0$ and is equal to zero on $\bar{R}_0$. Hence $\{u_m\}$ converges together with its derivatives to a function u uniformly on every compact subset of R, where u is harmonic in $R - \bar{R}_0$ and is equal to zero on $\bar{R}_0$. Hence

$$u_m \to u \text{ in } BD\text{-topology}.$$

Now we put

$$f = \psi - u$$

and

$$f_m = \psi - u_m.$$

Clearly $\{f_m\}$ converges to f in BD-topology and hence $f \in \boldsymbol{M}_\Delta(R)$. By Green's formula,

$$D_R(u, f_m) = 0.$$

From BD-convergence of $\{f_m\}$, we have

(5) $$D_R(u, f) = 0.$$

Next we put

$$\psi_n = w\varphi_n.$$

Since $\varphi_n \in \boldsymbol{M}_\Delta(R)$, there exists a sequence $\{\phi_{n,i}\}$ such that $\phi_{n,i} \in \boldsymbol{M}_0(R)$ and $\{\phi_{n,i}\}$ converges to φ_n in BD-topology for fixed n, as $i \to \infty$. By Green's formula

$$D_R(u, w\phi_{n,i}) = 0,$$

since $w\phi_{n,i} \in \boldsymbol{M}_0(R)$. Clearly the sequence $\{w\phi_{n,i}\}$ converges to $w\varphi_n$ in BD-topology, hence we have $\psi_n \in \boldsymbol{M}_\Delta(R)$ and

$$D_R(u, \psi_n) = 0.$$

The sequence $\{\psi_n\}$ and the function ψ satisfy the conditions in

Theorem 1. In fact, the condition (1) is clearly satisfied. For the condition (2), we note that $\{\varphi_n\}$ converges uniformly on R_1. Therefore we can assume $|\varphi_n|<M<+\infty$ on R_1, and we have the following inequality:

$$D_R(w\varphi_n)\leq D_R(\varphi_n)+2M\sqrt{D_R(w)D_R(\varphi_n)}+M^2D_R(w).$$

This shows that the sequence $\{D_R(\psi_n)\}$ is bounded. By Theorem 1, $\{d\psi_n\}$ converges to $d\psi$ weakly in $\Gamma(R)$. Thus we have

$$D_R(u, \psi)=0. \tag{6}$$

From the equality

$$D_R(u, \psi)=D_R(u, u)+D_R(u, f)$$

and (5), (6), we have

$$D_R(u, u)=0.$$

Hence u is equal to a constant. Since $u=0$ on R_0, $u=0$ on R. Thus $\psi=f\in \boldsymbol{M}_\Delta(R)$.

Remark: Theorem 2 is an extension of Proposition 10 in Royden [7] and Lemma 1.4.1 in Nakai [5].

Corollary. *$\boldsymbol{M}_\Delta(R)$ is complete with respect to BD-topology.*

Application: Let $\{G(z, w_n)\}$ be a sequence of Green's functions with poles w_n in R. Suppose that $\{w_n\}$ is a divergent sequence of points in R and that $\{G(z, w_n)\}$ converges to a harmonic function $h(z)$ uniformly on every compact subset in R. Then $h(z)$ is singular in the sense of Parreau [6] (Kuramochi [1]).

Proof. The following equality is well known:

$$D_R(\min[G(z, w_n), c])=2\pi c$$

for any positive number c. Clearly

$$\min[G(z, w_n), c]\in \boldsymbol{M}_\Delta(R)$$

and

$$\min[G(z, w_n), c]\rightarrow\min[h(z), c] \text{ in } B\text{-topology}.$$

Hence by Theorem 2

$$\min[h(z), c]\in \boldsymbol{M}_\Delta(R),$$

therefore

$$\lim_{R\ni z\rightarrow p}(\min[h(z), c])=0 \qquad \text{for any point } p\in\Delta,$$

where Δ is the harmonic boundary of R (cf. p. 185 in Nakai [4]). Let $u(z)$ be the greatest harmonic minorant of $h(z)$ and c. We have

$$0=\lim_{R\ni z\rightarrow p}\sup(\min[h(z), c])\geq \lim_{R\ni z\rightarrow p}\sup u(z)\geq 0$$

for any point $p\in\Delta$. By the maximum principle (Theorem 1.2, Corollary (a) p. 192 in Nakai [4]), $u(z)=0$ on R. Hence $h(z)$ is singular.

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