

35. Representation of a Semigroup by Row-Monomial Matrices over a Group

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Let G be a group written multiplicatively. An $n \times n$ matrix (where n can be any cardinal number) having at most one element of G in each row and zeros elsewhere is called a *row-monomial matrix* over G . The set $M(G, n)$ of all such matrices forms a semigroup under matrix multiplication. Schützenberger [2, 3] and Preston [1] have constructed *representations* of a semigroup S by row-monomial matrices, i.e., homomorphisms of S into $M(G, n)$. The purpose of this note is to present, without proofs, a new method for constructing such representations, which is more general than the methods used by Schützenberger and Preston.

Our method is similar to that used in the theory of monomial-representations of a group, and is somewhat analogous to the use, in ring theory, of modules over a ring R to construct representations of R by matrices over a field. We begin by defining the concept of a set with a semigroup S of operators (which, as in [4], we shall call an operand over S), and the endomorphisms of such sets (Section 1). In Section 2 we study a special class of operands, called free operands-with-zero, over a group G . These might be regarded as analogous to vector spaces. $M(G, n)$ is always isomorphic to the semigroup of endomorphisms of some free operand-with-zero over G . This leads in Section 3 to a procedure for determining all row-monomial representations of a semigroup. However, this result is not completely satisfactory, since it expresses the representations in terms of operands over S and their endomorphisms.

In Section 4, we restrict ourselves to a special kind row-monomial representation, viz., those in which at least one row can be “filled arbitrarily” [or “filled almost arbitrarily”]. This means that there is one row (say the i -th row) such that every monomial row vector [or every non-zero monomial row vector] actually occurs as the i -th row of one of the matrices corresponding to the elements of S . Thus the property in question is a kind of density condition.

It turns out that row monomial representations in which one row can be filled arbitrarily arise from strictly cyclic operands (in

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the sense of [4]). The author [4] has shown that such an operand consists essentially of the equivalence classes modulo a right congruence relation on S . This makes it possible to "internalize" the procedure of Section 3, expressing the representation in terms of right congruences on S and the multiplication of elements of S .

We note that the Schützenberger-Preston representations have the property that each row can be filled almost arbitrarily. The representations determined here include these as special cases. In Section 5, we present an example, which shows that the class of representations obtained in Section 4 is actually larger than the class of Schützenberger-Preston representations.

1. Operands and their endomorphisms. By an *operand* (or *right operand*) M_S over a semigroup S , we mean a set M together with a mapping $(x, a) \rightarrow xa$ of $M \times S$ into M , satisfying $x(ab) = (xa)b$. Operands over S may be identified in a fairly obvious way with representations of S by transformations, that is, homomorphisms of S into the semigroup of all transformations of a set M into itself. We shall also have occasion to consider *left operands* ${}_sM$, which are defined similarly except that we write elements of S on the left and assume $(ab)x = a(bx)$, and *bi-operands*, which involve a set M which is simultaneously a right S -operand and a left T -operand with the additional requirement that $(ax)b = a(xb)$ for all $x \in M$, $a \in T$, $b \in S$.

By an *endomorphism* of a (left or right) operand M we mean a mapping of M into itself which commutes with the operations of multiplying by elements of S . An endomorphism which is one-to-one and maps M onto itself will be called an *automorphism*.

2. Free operands. We call a left operand ${}_cM$ over a group G *free* if it has a *basis*, that is a subset $\{u_i\}$ such that each element of M can be written uniquely in the form gu_i for $g \in G$. It follows that each mapping of $\{u_i\}$ into M can be extended uniquely to an endomorphism of M . A free operand with one invariant element 0 adjoined will be called a *free operand-with-zero*. If M is a free operand-with-zero, we can define a mapping of the semigroup E of all those endomorphisms of M which leave 0 fixed into $M(G, n)$, by taking the image of an endomorphism π to be the matrix (a_{ij}) , where $a_{ij} = g$ if there exists $g \in G$ with $u_i\pi = gu_j$, and $a_{ij} = 0$ otherwise.

Theorem 1. *The mapping just defined is an isomorphism between E and $M(G, n)$.*

3. Row-monomial representations. By Theorem 1, we see that the construction of row-monomial representations of a semigroup S is equivalent to the construction of bi-operands ${}_cM_S$ where G is a group and ${}_cM$ is a free operand-with-zero. We can construct such bi-operands as follows. Choose an operand M_S over S , and an in-

variant element z of M_S . Let Σ be the group of all those automorphisms of M_S which leave z fixed. Choose a subgroup Δ of Σ , with the property that no element of Δ except the identity leaves fixed any element of M other than z . Define an equivalence σ on $M - \{z\}$ by: $x\sigma y$ if and only if $x\delta = y$ for some $\delta \in \Delta$. Suppose the number of σ -classes is n . Choose a family $\{u_i\}$ of elements of M , with exactly one u_i in each σ -class. Let G be the anti-isomorphic image of Δ under a correspondence π . For each $a \in S$, let $a\varphi$ be the $n \times n$ monomial matrix (a_{ij}) where: $a_{ij} = \delta\pi$, if $u_i a = u_j \delta$ for some (necessarily unique) $\delta \in \Delta$, and $a_{ij} = 0$, otherwise.

Theorem 2. *The mapping φ just constructed is a row-monomial representation of S . Moreover, every row-monomial representation of S can be constructed in this manner.*

4. We now present a method for constructing those row-monomial representations of S in which at least one row can be filled arbitrarily. To simplify our statements, we assume that S has an identity element 1. (This is no loss of generality, since an identity element can always be adjoined.) First, choose a right congruence σ on S . Let A be the set of all elements δ of S such that

- (1) $b\sigma c \iff (ab)\sigma(ac)$ for all $b, c \in S$, and
 - (2) a has a right inverse modulo σ , that is, $(ax)\sigma 1$ for some $x \in S$.
- Then T is a semigroup, the restriction of σ to T is a (two-sided) congruence, and the homomorphic image T/σ is a group. Now choose a subgroup G of T/σ such that
- (3) if a is an element of one of the σ -classes in G , and $(ab)\sigma b$ for some $b \in S$, then $a\sigma 1$.

Define an equivalence relation ρ on S by: $b\rho c$ if and only if $(ab)\sigma c$ for some a belonging to one of the σ -classes in G . Choose a family $\{a_i\}$ of n elements with exactly one in each ρ -class. For each $b \in S$, let $b\varphi$ be the $n \times n$ matrix (a_{ij}) , where $a_{ij} = \Sigma$ if Σ is a (necessarily unique) σ -class containing an element a such that $a_i b = a a_j$, and $a_{ij} = 0$ otherwise.

Theorem 3. *The mapping φ just constructed is a representation of S by row-monomial matrices, in which there are no zero rows and at least one row can be filled arbitrarily. Moreover, every such representation of S can be constructed in this manner.*

To take care of the case where zero-rows occur, a slight modification is necessary. We choose σ as before, but also choose one σ -class Z which is a right ideal. When choosing G we impose the additional requirement that $az \in Z$ whenever $z \in Z$ and a belongs to one of the σ -classes in G . The rest of the procedure is the same as before.

Theorem 4. *The mapping just constructed is a representation*

of S by row-monomial matrices, in which at least one row can be filled arbitrarily. Moreover, every such representation can be constructed in this manner.

5. Example. Suppose S is the direct product of a group H by the additive non-negative integers. Let us choose σ to be the identity relation on S , in which each class consists of a single element. Then $A = H \times 0$, that is, the set $\{(x, 0) : x \in H\}$. A/σ is A itself, and G can be taken to be any subgroup of A . Let us take $G = A$. Then each ρ -class consists of the elements of S having a common second coordinate. Suppose we choose $a_n = (1, n)$, where 1 is the identity of H , for $n = 0, 1, \dots$. Then, by the method of Theorem 3, the matrix (a_{ij}) corresponding to (h, n) is of infinite size, indexed by the non-negative integers, with $a_{ij} = h$ if $j = i + n$, and $a_{ij} = 0$ otherwise. The first row (but none of the other rows) can be filled almost arbitrarily. Thus we see that this representation cannot be found by the Schützenberger-Preston method. Indeed, it can be shown that the Schützenberger-Preston representation of S involves 1×1 matrices.

References

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