33. On the Representation of Large Even Integers as Sums of a Prime and an Almost Prime

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As is well known, the classical Goldbach problem, which remains still unsolved, is to prove that every even integer $\geqq 6$ is a sum of two odd prime numbers. In 1948 A. Rényi [3] proved that every even integer $\geqq 6$ is representable as a sum of a prime and an almost prime, i.e. an integer >1 with a bounded number of prime factors. Recently Ch.-D. Pan [2] sharpened this result by showing that every large even integer is a sum of a prime and an almost prime with at most five prime factors. Our aim in the present note is to prove the following theorem, which constitutes an improvement of this result of Pan:

Theorem. *Every sufficiently large even integer can be represented as a sum of a prime and an almost prime which is composed of at most four prime factors.*

Our proof of this theorem runs substantially on the same lines as in Pan [2].

By the same method we can also prove the following result: for every fixed integral value of $k\geqq 1$ there exist infinitely many primes p with $V(p+2k)\leqq 4$, where $V(m)$ denotes the total number of prime factors of m.

It is of some interest to note that Y. Wang [5] has proved under the extended Riemann hypotheses for Dirichlet L-functions that every sufficiently large even integer is representable as a sum of a prime and an almost prime with at most three prime factors and that for every fixed integral value of k there are infinitely many primes p with $V(p+2k)\leqq 3$.

1. We begin with reproducing the Fundamental Theorem of Pan [2] in a slightly refined form.

Let k and l be two integers such that $k\geqq 1$, $0\leqq l\leqq k-1$, $(k, l)=1$. We define[1] for $x\geqq 2$

$$P(x, k, l)=\sum_{\substack{p\leq x\\ p\equiv l \pmod{k}}} \alpha_p,$$

where

$$\alpha_p=\log p\ \exp\left(-p\frac{\log x}{x}\right),$$

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1) Throughout in this note the letters p and q are used to represent prime numbers.

and put

$$R_k(x)=P(x,k,l)-\frac{x}{\varphi(k)\log x}.$$

Let χ be a character defined to the modulus k and let $L(s,\chi)$ $(s=\sigma+it)$ denote the corresponding Dirichlet L-series. By a result of D. A. Burgess [1, Theorem 2], we find that if χ is a non-principal character (mod k) then

$$\left|L\left(\frac{1}{2}+it,\chi\right)\right|<c(|t|+1)k^{\frac{7}{32}+\delta}$$

for any $\delta>0$, $c=c(\delta)$ being a positive constant. It is easily seen from the proof of Pan [2] that his Fundamental Theorem can now be stated as follows: for any fixed $\varepsilon>0$ and any $A\geqq 3$ we have

$$\sum_{m\leqq x^{\frac{8}{23}-\varepsilon}}|\mu(m)2^{v(m)}R_m(x)|=O\left(\frac{x}{\log^A x}\right),$$

where $v(m)$ denotes the number of distinct prime factors of m and where (and in what follows) the O-constant may depend only on ε and A.

2. Let ε be a sufficiently small fixed positive real number and let $N\geqq N_0=N_0(\varepsilon)$ be a large even integer. We set for $u\geqq 2$

$$P(N^{\frac{1}{u}})=\sum\alpha_p,$$

the summation being extended over the set of primes $p\leqq N$, $(p,N)=1$, such that $N-p$ is not divisible by any prime $q\leqq N^{\frac{1}{u}}$. We wish to estimate $P(N^{\frac{1}{u}})$ for some values of u.

For the sake of brevity let us put $A_N=(e^C C_N)^{-1}$, where C denotes the Euler constant and

$$C_N=\prod_{\substack{p/N\\p>2}}\frac{p-2}{p-1}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}.$$

Now we take h so as to satisfy

$$\frac{3\cdot 20}{391}+\frac{2.20}{391(h-1)}=\frac{8}{23}-\varepsilon,$$

i.e. $h=(116-391\varepsilon)/(76-391\varepsilon)$. Then for $x\geqq x_0=x_0(\varepsilon)$

$$0.4229>\tau\overset{\text{def}}{=}\log(h+\varepsilon)>\sum_{\substack{x<p\leqq x^h\\(p,N)=1}}\frac{1}{\varphi(p)},$$

$$1.5264>\lambda\overset{\text{def}}{=}h+\varepsilon>\prod_{\substack{x<p\leqq x^h\\(p,N)=1}}\left(1-\frac{1}{\varphi(p)}\right)^{-1}.$$

We obtain by the sieve method of Viggo Brun, via the Fundamental Theorem of Pan, that

$$P(N^{\frac{20}{391}})>39.1(1-c)A_N\frac{N}{\log^2 N}+O\left(\frac{N\log\log N}{\log^3 N}\right),$$

where

$$c=\sum_{m=0}^{\infty}\frac{\lambda^{m+1}((m+1)\tau)^{2m+4}}{(2m+4)!}<0.004379.$$

Hence

$$(1)\qquad P(N^{\frac{20}{391}})>38.9287\,A_N\frac{N}{\log^2 N}+O\Big(\frac{N\log\log N}{\log^3 N}\Big).$$

3. Let v be a real number satisfying $4\leqq v<391/20$, and let q be a fixed prime number in the interval $N^{\frac{1}{v}}<q\leqq 2N^{\frac{1}{v}}$. We define

$$S_v(q)=\sum \alpha_p,$$

where the summation is extended over the set of primes $p\leqq N$, $(p, N)=1$, such that $N-p\not\equiv 0 \pmod{p'}$ for any $p'\leqq N^{\frac{1}{v}}$ and $N-p\equiv 0 \pmod{q}$. To estimate $S_v(q)$ we apply the sieve method of A. Selberg (cf. [4, Appendix]). Thus, if q is a non-exceptional prime[2)] then we get, again by the Fundamental Theorem of Pan,

$$S_v(q)\leqq A_N\frac{C_\varepsilon(v, t_q)}{\varphi(q)}\frac{N}{\log^2 N}+O\Big(\frac{N(\log\log N)^3}{\varphi(q)\log^3 N}\Big)$$

where $t_q=(\log N)/\log q$ and

$$C_\varepsilon(v, t)=\begin{cases}\dfrac{2e^C v}{a} & (0<a\leqq 1),\\[2mm] \dfrac{2e^C v}{2a-1-a\log a} & (1<a\leqq 2.9),\end{cases}$$

$$a=\frac{v}{2}\Big(\frac{8}{23}-2\varepsilon-\frac{1}{t}\Big).$$

On the basis of this fact it is not difficult to prove that

$$P(N^{\frac{20}{391}})-P(N^{\frac{1}{8}})\leqq J(\varepsilon)A_N\frac{N}{\log^2 N}+O\Big(\frac{N\log\log N}{\log^{5/2} N}\Big),$$

where

$$J(\varepsilon)=\int_8^{\frac{391}{20}}\frac{C_\varepsilon(v, v)}{v}dv.$$

We shall show that for some sufficiently small $\varepsilon>0$ we have

$$(2)\qquad J(\varepsilon)\leqq 28.5753.$$

Now, substituting $s=\dfrac{v}{2}\Big(\dfrac{8}{23}-2\varepsilon-\dfrac{1}{v}\Big)$ and putting $\varepsilon=0$, we get

$$J(0)=\frac{23e^C}{2}\Big(\int_{\frac{41}{46}}^{1}\frac{ds}{s}+\int_1^{\frac{29}{10}}\frac{ds}{2s-1-s\log s}\Big)$$
$$=\frac{23e^C}{2}(J_1+J_2),$$

say. We have

$$J_1=\log\frac{46}{41}<0.115070.$$

To estimate J_2, we note that the function $f(s)=1/(2s-1-s\log s)$ is convex for $1\leqq s\leqq 2.9$. Hence

$$J_2\leqq 0.05(f(1)+f(2.9))+0.1(f(1.1)+f(1.2)+\cdots+f(2.8)).$$

2) For the meaning of an 'exceptional' prime we refer to [3, Lemma 1].

We evaluated J_2 in this way using the electronic computer, HIPAC 103, and found that

$$J_2 < 1.280044.$$

Thus we have $J(0) < 28.575216$, and the result (2) follows from this by continuity argument.

It now follows from (1) that

$$S_1 \overset{\text{def}}{=} P(N^{\frac{1}{8}}) > 10.3534 A_N \frac{N}{\log^2 N} + O\left(\frac{N \log\log N}{\log^{5/2} N}\right). \tag{3}$$

4. We put

$$S_2 = \sum \alpha_p,$$

the summation being extended over the set of primes $p \leqq N$, $(p, N) = 1$, such that $N - p \not\equiv 0 \pmod{p'}$ for any prime $p' \leqq N^{\frac{1}{8}}$ and $N - p \equiv 0 \pmod{q}$ for at least three distinct primes q, $(q, N) = 1$, in the interval $N^{\frac{1}{8}} < q \leqq N^{\frac{1}{4}}$. Then we have, as in the preceding section,

$$3S_2 \leqq K(\varepsilon) A_N \frac{N}{\log^2 N} + O\left(\frac{N \log\log N}{\log^{5/2} N}\right),$$

where

$$K(\varepsilon) = \int_4^8 \frac{C_\varepsilon(8, v)}{v} dv$$

and $K(\varepsilon) \leqq 31.0584$ for some sufficiently small $\varepsilon > 0$ since we have

$$K(0) = \frac{23 e^C}{2} \log \frac{41}{9} < 31.058372.$$

Hence

$$S_2 \leqq 10.3528 A_N \frac{N}{\log^2 N} + O\left(\frac{N \log\log N}{\log^{5/2} N}\right). \tag{4}$$

5. Now, let us fix $\varepsilon > 0$ so small that the inequalities (3) and (4) hold simultaneously, and suppose that $N \geqq N_0 = N_0(\varepsilon)$ be a sufficiently large even integer. We consider the sum

$$S = \sum \alpha_p,$$

where the summation is extended over the set of primes $p \leqq N$, $(p, N) = 1$, such that $N - p \not\equiv 0 \pmod{p'}$ for any prime $p' \leqq N^{\frac{1}{8}}$, $N - p \equiv 0 \pmod{q}$ for at most two primes q, $(q, N) = 1$, in $N^{\frac{1}{8}} < q \leqq N^{\frac{1}{4}}$, and $N - p \not\equiv 0 \pmod{q^2}$ for any prime q in $N^{\frac{1}{8}} < q \leqq N^{\frac{1}{4}}$. It is clear that the sum S_3 of the α_p taken over the set of primes $p \leqq N$, $(p, N) = 1$, such that $N - p \not\equiv 0 \pmod{p'}$ for any $p' \leqq N^{\frac{1}{8}}$ and $N - q \equiv 0 \pmod{q^2}$ for some q, $(q, N) = 1$, in $N^{\frac{1}{8}} < q \leqq N^{\frac{1}{4}}$ is of $O(N^{\frac{7}{8}})$. By (3) and (4) we thus obtain

$$\begin{aligned} S &\geqq S_1 - S_2 - S_3 \\ &> (10.3534 - 10.3528) A_N \frac{N}{\log^2 N} + O\left(\frac{N \log\log N}{\log^{5/2} N}\right) \\ &> 0.0005 A_N \frac{N}{\log^2 N} > 1. \end{aligned} \tag{5}$$

It follows from (5) that there exists at least one prime $p \leqq N - 3$, $(p, N) = 1$, such that $V(N - p) \leqq 4$, provided that $N \equiv 0 \pmod{2}$ be large enough. Since $N=p+(N-p)$, this completes the proof of our theorem.

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