32. *On Differentially Integral Elements*

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1. Let R be a differential ring with a derivation δ, and P be a differential subring of R. J. Brzeziński [1] has introduced the following notions. An element x of R is called *differentially integral* with respect to P if there exists a finitely generated P-submodule of R containing all the derivatives: $\delta^0 x=x$, δx, $\delta^2 x=\delta\cdot\delta x, \cdots$. When R is moreover an integral domain, R is said to be *differentially-integrally closed* if every element of the quotient field $Q(R)$ of R which is differentially integral with respect to R is contained in R.

If one wishes to go into the differential algebra of non-zero characteristic, it will be found that these notions may not sufficiently answer the purpose.[1] In a recent paper [3], K. Okugawa, taking up a differential ring with Hasse's *higher differentiations*, has developed the Picard-Vessiot theory for linear homogeneous differential equations, which is a generalization of the work of E. Kolchin [2] to the case of an arbitrary characteristic.

In this note, we consider the corresponding notion of the differentially integral element concerning the differential ring in the latter sense, and we generalize the condition which is given in [1], for a unique factorization domain to be differentially-integrally closed. This we discuss in a manner similar to that of [1].

2. The notation and terminology will be as in [3]. Let R be a ring.[2] A sequence $\boldsymbol{\delta}=\{\delta_\nu;\ \nu=0, 1, 2, \cdots\}$ of maps $\delta_\nu\colon R\to R$ is called a *differentiation* in R, if it satisfies

(D1) $\delta_0 x=x$, (D2) $\delta_\nu(x+y)=\delta_\nu x+\delta_\nu y$, for all ν,

(D3) $\delta_\nu(x\cdot y)=\sum_{\lambda+\mu=\nu}\delta_\lambda x\cdot\delta_\mu y$, for all ν, (D4) $\delta_\lambda(\delta_\mu x)=\binom{\lambda+\mu}{\lambda}\delta_{\lambda+\mu}x$, for all λ, μ.

A ring $\boldsymbol{R}$ with the mutually commutative differentiations $\boldsymbol{\delta}_i=\{\delta_{i,\nu};\ \nu\geqq 0\}$ $(1\leqq i\leqq m)$ preassigned in $\boldsymbol{R}$ is called a *differential ring*.[3] The set $\Theta=\{\delta_{1,\nu_1}\cdots\delta_{m,\nu_m};\ \nu_1\geqq 0, \cdots, \nu_m\geqq 0\}$ is regarded as the domain of differential operators in $\boldsymbol{R}$.

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1) In the case of the characteristic $p>0$, since $(1/x)^p$ is differentially integral with respect to R for any $x\in R$, we see at once that R is never differentially-integrally closed except for the trivial case where R itself is a field.

2) When we speak of a *ring* in this note, we always suppose tacitly that it is a commutative ring with unity.

3) In case of a characteristic zero, if $\boldsymbol{R}$ is an integral domain containing a field then it is regarded as the usual differential ring, since $\delta_{i,\nu}=\frac{1}{\nu!}\delta_{i,1}$ and $\delta_{i,1}$ is the usual derivation in $\boldsymbol{R}$ for $1\leqq i\leqq m$. See [3], §1.

Definition. *Let $\boldsymbol{P}$ be a differential subring of $\boldsymbol{R}$. An element $x \in \boldsymbol{R}$ is called differentially integral with respect to $\boldsymbol{P}$ if there exists a finitely generated $\boldsymbol{P}$-submodule of $\boldsymbol{R}$ containing $\theta \cdot x$ for all $\theta \in \Theta$.*

The meaning of the above definition may be clearer in the special case where $\boldsymbol{P}$ is noetherian. Let $\boldsymbol{P}$ be noetherian, then since finitely generated $\boldsymbol{P}$-modules satisfy the ascending chain condition, the chain

$$(x) \subset (x, \theta^{(1)}x) \subset (x, \theta^{(1)}x, \theta^{(2)}x)^{4)} \subset \cdots, \qquad \theta^{(i)} \in \Theta (i=1, 2, \cdots)$$

must stop. Thus there exist a finite number of operators $\theta^{(1)}, \cdots, \theta^{(N)}$ such that

$$(1) \qquad \theta x \in (x, \theta^{(1)}x, \cdots, \theta^{(N)}x) \qquad \text{for all } \theta \in \Theta.$$

It is readily seen, conversely, that if there exist $\theta^{(1)}, \cdots, \theta^{(N)}$ for which (1) holds, then x is differentially integral with respect to $\boldsymbol{P}$. By (1) we see that x can be said to satisfy (simultaneous) linear homogeneous (partial) differential equations of *finite type* with coefficients in $\boldsymbol{P}$, which is referred to in [3], §5. Thus we have

Theorem 1. *If $\boldsymbol{P}$ is noetherian, then x is differentially integral with respect to $\boldsymbol{P}$ if and only if x satisfies linear homogeneous differential equations of finite type over $\boldsymbol{P}$.*

We denote by $C_{\boldsymbol{R}}(\boldsymbol{P})$ the set of all the elements of $\boldsymbol{R}$ which are differentially integral with respect to $\boldsymbol{P}$, i.e. the closure of $\boldsymbol{P}$ in $\boldsymbol{R}$.

Theorem 2. *$C_{\boldsymbol{R}}(\boldsymbol{P})$ is a differential subring of $\boldsymbol{R}$, containing $\boldsymbol{P}$.*

Proof. Let $x, y (\in \boldsymbol{R})$ be differentially integral with respect to $\boldsymbol{P}$ then there exist $\boldsymbol{P}$-submodules $M=(r_1, \cdots, r_k)$ and $N=(s_1, \cdots, s_l)$ of $\boldsymbol{R}$ such that $\theta x \in M$ and $\theta y \in N$ for all $\theta \in \Theta$, respectively. Using (D2) and (D3) repeatedly, we have $\theta(x+y)=\theta x+\theta y$ and $\theta(x \cdot y)=\sum \theta' x \cdot \theta'' y$,[5] respectively. Hence we see that $\theta(x+y)$ and $\theta(x \cdot y)$ belong to the finitely generated modules $M+N$ and $(\cdots, r_i \circ s_j, \cdots)$, respectively.

Finally, let θ_1, θ_2 be any two operators in Θ. Using (D4) repeatedly, it can be verified that there exist an operator θ_3[6] and an integer c such that $\theta_2(\theta_1 x)=c\theta_3 x$. Hence all derivatives of $\theta_1 x$ are contained in M, which implies $C_{\boldsymbol{R}}(\boldsymbol{P})$ is stable under differential operators. q.e.d.

3. From now on we consider the case where $\boldsymbol{R}$ is an integral domain, whose quotient field will be denoted by $Q(\boldsymbol{R})$, the characteristic p of $Q(\boldsymbol{R})$ being arbitrary. The fact that the structure of the differential field of $Q(\boldsymbol{R})$ is induced uniquely by the given structure of $\boldsymbol{R}$ is referred to in [3], Proposition 2.3.

Definition. *$\boldsymbol{R}$ is said to be differentially-integrally closed if $C_{Q(\boldsymbol{R})}(\boldsymbol{R})=\boldsymbol{R}$.*

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4) $\boldsymbol{P}$-module generated by $x, y, \cdots$ is denoted by $(x, y, \cdots)$.

5) For $\theta=\delta_{1,\nu_1} \cdots \delta_{m,\nu_m}$, let $\mathrm{ord}.\theta=(\nu_1, \cdots, \nu_m)$, regarded as a vector over integers, then the summation runs over all θ', θ'' such that $\mathrm{ord}.\theta'+\mathrm{ord}.\theta''=\mathrm{ord}.\theta$.

6) $\mathrm{ord}.\theta_3=\mathrm{ord}.\theta_1+\mathrm{ord}.\theta_2$.

Theorem 3. *Assume that* $\boldsymbol{R}$ *is a differential domain containing a field,*[7] *and assume further that in* $\boldsymbol{R}$ *the unique factorization theorem for elements holds true. Then* $\boldsymbol{R}$ *is differentially-integrally closed if and only if for any irreducible element* $x \in \boldsymbol{R}$ *there exists* $\theta \in \Theta$ *such that* θx *is not divisible by* x *in* $\boldsymbol{R}$.

The proof will be preceded by the following lemma.

Lemma. *Assume* R *to be an integral domain containing a field,* $\delta = \{\delta_\nu;\ \nu \geqq 0\}$ *to be a differentiation in* R. *Let* $x \in R$. *In the case where* $\delta_\nu x$ *is divisible by* x *for all* ν,

(I) $$\delta_\nu(1/x)\cdot x \in R \quad \text{for all } \nu.$$

In the case where there exists λ *such that* $\delta_\lambda x$ *is not divisible by* x, *let* λ_0 *be the smallest among such* λ, *then for all* ν

(II, ν) $$\delta_{\nu\lambda_0}(1/x) = [(-1)^\nu(\delta_{\lambda_0}x)^\nu + x\cdot r]/x^{\nu+1},\ r \in R,$$

(III, ν) $$\delta_\mu(1/x)\cdot x^{\nu+1} \in R \quad \text{for } \mu < (\nu+1)\lambda_0.$$

Proof. Operating δ_ν on both sides of $x\cdot(1/x)=1$, we see by (D3) that

(2) $$\delta_\nu(1/x)\cdot x + \sum_{\lambda=0}^{\nu-1}\delta_\lambda(1/x)\cdot\delta_{\nu-\lambda}x = 0.$$

From this (I) is proved inductively with respect to ν. In fact, in that case $\delta_{\nu-\lambda}x$ is divisible by x and by the induction assumption $\delta_\lambda(1/x)\cdot x \in R$ for $0 \leqq \lambda \leqq \nu-1$.

As for (II), (III), it is noted that $\lambda_0=1$ when $p=0$,[8] and that λ_0 is a power of p when $p>0$. The latter is seen as follows. Let $\rho = c_0 + c_1 p + \cdots + c_e p^e$ be the p-adic expression of a positive integer ρ, then by (D4) $\delta_1^{c_1}\cdots\delta_{p^e}^{c_e} = \gamma\cdot\delta_\rho$ where $\gamma = \rho!/[(p!)^{c_1}\cdots(p^e!)^{c_e}] \not\equiv 0\ (p)$.[9] Therefore, if each of $\delta_1 x, \cdots, \delta_{p^e}x$ is divisible by x, then $\delta_\rho x$ is also divisible by x, which means that the smallest λ_0 for which $\delta_{\lambda_0}x$ is not divisible by x is a power of p.

We proceed in the following order: we show first (II, ν) $\Rightarrow$ (III, ν) and then (II, ν), (III, $\nu-1$), and (III, ν) $\Rightarrow$ (II, $\nu+1$).

(II, ν) $\Rightarrow$ (III, ν). We may assume $\nu\lambda_0 \leqq \mu < (\nu+1)\lambda_0$. Putting $\mu = \nu\lambda_0 + c$, $0 \leqq c < \lambda_0$, we shall prove this inductively with respect to c. When $c=0$ our assertion is evidently true, and therefore in particular it is so when $p=0$ —— consequently $\lambda_0=1$ by the above. Thus we may assume $p>0$.

By (D4), $\delta_c\cdot\delta_{\nu\lambda_0} = \binom{\nu\lambda_0+c}{c}\cdot\delta_\mu$ and further $\binom{\nu\lambda_0+c}{c} \not\equiv 0\ (p)$ since λ_0 is a power of p and $c<\lambda_0$.[10] Therefore δ_μ is decomposed into two operators: $\delta_\mu = \beta\delta_c\cdot\delta_{\nu\lambda_0}$ where β is an integer.

Now, by (II, ν) evidently $\delta_{\nu\lambda_0}(1/x)\cdot x^{\nu+1} \in R$. Operating δ_c on this, we have

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7) i.e. $\boldsymbol{R}$ is either a Ritt algebra or a differential domain of non-zero characteristic.

8) See 3).

9) See [3], §1, p 297.

10) See also [3], §1, p 297.

$$(3)\qquad [\delta_c\cdot\delta_{\nu\lambda_0}(1/x)]\cdot x^{\nu+1}+\sum_{\substack{c'+c''=c\\ c''>0}}[\delta_{c'}\cdot\delta_{\nu\lambda_0}(1/x)]\cdot\delta_{c''}(x^{\nu+1})\ \in R.$$

By the induction assumption $[\delta_{c'}\cdot\delta_{\nu\lambda_0}(1/x)]\cdot x^{\nu+1}\in R$ for $0\leqq c'<c$. By the definition of λ_0, for $0<c''\leqq c$ $\delta_{c''}(x^{\nu+1})$ is divisible by $x^{\nu+1}$ since c is not greater than λ_0. Therefore it follows from (3) that $[\delta_c\cdot\delta_{\nu\lambda_0}(1/x)]\cdot x^{\nu+1}\in R$, which was to be proved.

(II, ν), (III, $\nu-1$), and (III, ν) $\Rightarrow$ (II, $\nu+1$). Let the characteristic p be arbitrary. Substituting $(\nu+1)\lambda_0$ for ν in (2), we have

$$(4)\qquad \delta_{(\nu+1)\lambda_0}(1/x)=-(1/x)\{\sum_{\substack{\mu+\mu'=(\nu+1)\lambda_0\\ \mu'>0}}\delta_\mu(1/x)\cdot\delta_{\mu'}x\}$$

$$=-(1/x)\{\sum_{\substack{\mu+\mu'=(\nu+1)\lambda_0\\ \mu<\nu\lambda_0}}\delta_\mu(1/x)\cdot\delta_{\mu'}x+\delta_{\nu\cdot\lambda_0}(1/x)\cdot\delta_{\lambda_0}x+\sum_{\substack{\mu+\mu'=(\nu+1)\lambda_0\\ \nu\cdot\lambda_0<\mu<(\nu+1)\lambda_0}}\delta_\mu(1/x)\cdot\delta_{\mu'}x\}.$$

The following relations (5), (6), and (7) are immediate consequences of (III, $\nu-1$), (II, ν), and (III, ν) respectively. In (7) we also use the fact that $\delta_{\mu'}x$ is divisible by x for $\mu'<\lambda_0$.

$$(5)\qquad \delta_\mu(1/x)\cdot\delta_{\mu'}x=r_\mu/x^\nu,\quad r_\mu\in R,\quad \text{for } \mu<\nu\lambda_0.$$

$$(6)\qquad \delta_{\nu\lambda_0}(1/x)\cdot\delta_{\lambda_0}x=[(-1)^\nu(\delta_{\lambda_0}x)^{\nu+1}+x\cdot s]/x^{\nu+1},\ s\in R.$$

$$(7)\qquad \delta_\mu(1/x)\cdot\delta_{\mu'}x=t_\mu/x^\nu,\quad t_\mu\in R,\quad \text{for } \nu\lambda_0<\mu<(\nu+1)\lambda_0,\quad 0<\mu'<\lambda_0.$$

Combining (4), (5), (6), and (7), we have $\delta_{(\nu+1)\lambda_0}(1/x)=[(-1)^{\nu+1}(\delta_{\lambda_0}x)^{\nu+1}-x(\sum r_\mu+s+\sum t_\mu)]/x^{\nu+2}$, which was to be proved. q.e.d.

Proof of Theorem 3. First, we assume that there exists an irreducible element $x\in\boldsymbol{R}$ by which $\theta\cdot x$ is divisible for every $\theta\in\Theta$. According to Lemma (I), $\delta_{i,\nu}(1/x)\in(1/x)\cdot\boldsymbol{R}$ for all i, ν. From this it follows immediately that $\theta(1/x)\in(1/x)\cdot\boldsymbol{R}$ for all $\theta\in\Theta$, as Θ is generated by $\delta_{i,\nu}(1\leqq i\leqq m,\ \nu\geqq 0)$. Therefore by definition $1/x$ is differentially integral with respect to $\boldsymbol{R}$. On the other hand, x being irreducible, $1/x$ is not contained in $\boldsymbol{R}$. Thus we see that in this case $\boldsymbol{R}$ is never differentially-integrally closed.

Next, we shall prove the converse, namely that if, in $\boldsymbol{R}$, for any irreducible x there exists θ such that θx is not divisible by x, then $\boldsymbol{R}$ is differentially-integrally closed. If we assume the contrary, then there would be x, $y\in\boldsymbol{R}$ for which y/x is differentially integral with respect to $\boldsymbol{R}$, and y/x is not contained in $\boldsymbol{R}$.

As $(y/x)\cdot z$ is also differentially integral with respect to $\boldsymbol{R}$ for $z\in\boldsymbol{R}$ (see Theorem 2), without loss of generality we can furthermore assume x is irreducible. Then by the above assumption for $\boldsymbol{R}$ there exists $\theta\in\Theta$ such that θx is not divisible by x. From this it follows readily that there exist i, ν such that $\delta_{i,\nu}x$ is not divisible by x, as Θ is generated by $\delta_{i,\nu}(1\leqq i\leqq m,\ \nu\geqq 0)$. Now, fixing such an i, let us write for simplicity $\delta_{i,\nu}=\delta_\nu$. Let λ_0 be, as in (II), (III) of Lemma the smallest for which $\delta_{\lambda_0}x$ is not divisible by x.

From the fact that y/x is differentially integral with respect to $\boldsymbol{R}$ it follows that there exists $k\in\boldsymbol{R}$ such that $k\neq 0$ and $k\cdot\theta(y/x)\in\boldsymbol{R}$ for all $\theta\in\Theta$, since all the $\theta(y/x)$ are contained in a finitely generated

$\boldsymbol{R}$-module. Hence, in particular, $k\cdot\delta_\nu(y/x)\in\boldsymbol{R}$ for all ν.

We shall show that k is divisible by x^ν for any integer ν, which will contradict the unique factorization theorem in $\boldsymbol{R}$. As $k\cdot\delta_{0\lambda\nu}(y/x)\in\boldsymbol{R}$ for all ν,

$$k\cdot\delta_{\nu\cdot\lambda_0}(1/x)\cdot y+\sum_{\substack{\mu+\mu'=\nu\lambda_0\\ \mu<\nu\lambda_0}}k\cdot\delta_\mu(1/x)\cdot\delta_{\mu'}x\in\boldsymbol{R}.$$

Using (II), (III) of Lemma here, we have

$$k\cdot y\cdot[(-1)^\nu(\delta_{\lambda_0}x)^\nu+x\cdot r]/x^{\nu+1}+k\cdot u/x^\nu\in\boldsymbol{R},\quad\text{where } r,\ u\in\boldsymbol{R}.$$

Since y and $(\delta_{\lambda_0}x)^\nu$ are both relatively prime to x, from this we can prove inductively with respect to ν that k is divisible by x^ν for all ν.

References

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