55. On E. Lindelöf's Theorem on the Meromorphic Function of Bounded Characteristic in the Unit Circle

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1. **Introduction.** E. Lindelöf's theorem on asymptotic values is not always true for the meromorphic function of bounded characteristic. Indeed, putting $f(z)=(1-z)\exp\{(1+z)/(1-z)\}$, $f(z)$ is regular and of bounded characteristic in $|z|<1$, because $f(z)$ is the quotient of two bounded regular functions $(1-z)$ and $\exp\{-(1+z)/(1-z)\}$. Then we have easily

$$\lim_{\theta\to\pm 0} f(e^{i\theta})=0 \quad \text{and} \quad \lim_{r\to 1-0} f(r)=\infty,$$

which shows that E. Lindelöf's theorem is not true for $f(z)$.

The object of this note is to give the decisive answer to the question in what form E. Lindelöf's theorem should be modified in the case of the meromorphic function with bounded characteristic in $|z|<1$.

2. **Theorem 1.** Let $f(z)$ be the meromorphic function of bounded characteristic in the unit disk D, P a point on the unit circle C, and A a Jordan arc contained in $D\smile C$ and terminating at P. We denote by $D(A_1, A_2, \varepsilon)$ the domain bounded by the periphery of $U(P, \varepsilon)$*) and two Jordan arcs A_1, A_2 having no common point except for P.

Our main theorem is

Theorem 1. *Let $f(z)$ be meromorphic and of bounded characteristic in D. If $f(z)$ tends to a_i as $z\to P$ along A_i $(i=1, 2)$, then following alternatives are possible:*

(1) $a_1=a_2$ *and $f(z)$ tends uniformly to $a_1=a_2$ as $z\to P$ in* $\overline{D}(A_1, A_2, \varepsilon)$,

or

(2) *Picard's exceptional value in $D(A_1, A_2, \varepsilon)$ distinct from a_i $(i=1, 2)$ is at most one.*

Remark 1. Applying the theorem of Iversen-Gross ([3] p. 24) to $D(A_1, A_2, \varepsilon)$, we can conclude that following alternatives are possible:

(1) $a_1=a_2$ and the cluster set at P reduces to this single point,

or

(2) every value distinct from a_i $(i=1, 2)$, *except for at most two values*, is taken infinitely often by $f(z)$ in $D(A_1, A_2, \varepsilon)$.

Hence, Theorem 1 means that the boundedness of characteristic yields the reduction of number of exceptional values in $D(A_1, A_2, \varepsilon)$.

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*) $U(P, \varepsilon)$ is the ε-neighborhood of P.

Remark 2. $f(z)=(1-z)\exp\{(1+z)/(1-z)\}$ shows that (2) of Theorem 1 is best possible. Indeed, $w=f(z)$ is the regular function of bounded characteristic in D, whose Picard's exceptional values in D contain evidently $w=0$ and ∞. On the other hand, $\lim_{\theta\to\pm 0} f(e^{i\theta})=0$ and $\lim_{r\to 1-0} f(r)=\infty$. Putting $A_1=E(e^{i\theta}:\theta>0)$ and $A_2=E(e^{i\theta}:\theta<0)$, Picard's exceptional value in $D(A_1, A_2, \varepsilon)$ distinct from $w=0$ is exactly one value, i.e. $w=\infty$, because if $w=\alpha$ ($\neq 0, \infty$) is omitted in $D(A_1, A_2, \varepsilon)$, then $f(z)$ omits three distinct values $w=0, \infty, \alpha$ in $D(A_1, A_2, \varepsilon)$ and by the theorem of Lindelöf-Iversen-Gross ([3] p. 5) $f(z)$ tends uniformly to 0 as $z\to 1$ in $\overline{D}(A_1, A_2, \varepsilon)$, which is contrary to $\lim_{r\to 1-0} f(r)=\infty$.

3. Theorems 2 and 3. These theorems yield examples having no Picard's exceptional value of Theorem 1, (2).

Theorem 2. *There exists a meromorphic function $f(z)$ of bounded characteristic in D represented by the quotient of two infinite Blaschke products such that, for fixed ϑ $(0<\vartheta<\pi/2)$,*

(1) $\lim_{z\to 1,\, \arg(1-z)=-\vartheta} f(z)=0$ *and* $\lim_{z\to 1,\, \arg(1-z)=+\vartheta} f(z)=\infty$.

(2) *$f(z)$ takes every value, without any exception, infinitely many times, in the sector:* $|\arg(1-z)|\leqq\vartheta$.

Theorem 3. *There exists $f(z)$ regular and of bounded characteristic in D such that, for any positive constant $\varepsilon\,(<\pi/2)$,*

(1) $\lim_{z\to 1,\, |\arg(1-z)|=\varepsilon} f(z)=\infty$.

(2) *$f(z)$ takes every finite value, without any exception, infinitely many times in the sector:* $|\arg(1-z)|<\varepsilon$, *i.e.* $\arg(z)=0$ *is Julia-line.*

Remark 3. In the preceding paper ([5], p. 472), the author constructed the quotient $f(z)$ of two infinite Blaschke products such that

(1) $f(z)$ has infinite number of zeros and poles on $\arg(1-z)=-\vartheta$ and $\arg(1-z)=+\vartheta$ respectively $(0<\vartheta<\pi/2)$.

(2) $\lim_{z\to 1,\, \arg(1-z)=-\vartheta} f(z)=0$, and $\lim_{z\to 1,\, \arg(1-z)=+\vartheta} f(z)=\infty$.

We can prove that $f(z)$ has the properties desired in Theorem 2.

Remark 4. Setting $f(z)=B(z)\exp\{(1+z)/(1-z)\}$, where $B(z)=\prod_{n=1}^{+\infty}(a_n-z)/(1-a_n z)$ $(a_n=1-1/n^2)$, $f(z)$ is the regular function of bounded characteristic in D, because $f(z)$ is the quotient of two bounded regular functions $B(z)$ and $\exp\{-(1+z)/(1-z)\}$. It can be shown that $f(z)$ has the properties desired in Theorem 3.

4. Outline of proofs. Theorems 1 and 2 are proved by the systematic use of the harmonic majorant, which was effectively used by C. Tanaka [4] and F. W. Gehring [2] in the case of the regular function with bounded characteristic in D.

Theorem 3 can be proved by the detailed study of $B(z)=\prod_{n=1}^{+\infty}(a_n-z)/(1-a_n z)$ $(a_n=1-1/n^2)$ due to F. Bagemihl and W. Seidel ([1], pp. 7–9). We shall give their full proofs in another journal in the near future.

References

[1] F. Bagemihl and W. Seidel: Sequential and continuous limits of meromorphic functions. Ann. Acad. Sci. Fenn. I. A., **280**, 1–16 (1960).

[2] F. W. Gehring: The asymptotic values for analytic functions with bounded characteristic. Quarterly Jour. Math., **9**, 282–289 (1958).

[3] K. Noshiro: Cluster Sets. Springer, Berlin (1960).

[4] C. Tanaka: An extension of Kintchine-Ostrowski's theorem and its some applications. Kōdai Math. Sem. Rep., **9**, 97–104 (1957).

[5] ——: On the boundary values of Blaschke products and their quotients. Proc. Amer. Math. Soci., **14**, 472–476 (1963).