70. *Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. X*

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In this paper we shall treat of some applications of Theorems 2 and 3 established in the first paper of the same title as above [1].

Definitions and preliminaries. Let M be an arbitrarily prescribed positive constant; let $\{\lambda_\nu^{(\omega)}\}_{\nu=1,2,3,\dots}$ be any infinite sequence of complex numbers with multiplicities properly counted such that $\sup_\nu |\lambda_\nu^{(\omega)}| \leqq M$; let c_ω be any finite complex number, not zero; let $\{\varphi_\nu^{(\omega)}\}_{\nu=1,2,3,\dots}$ and $\{\psi_\mu^{(\omega)}\}_{\mu=1,2,3,\dots}$ both be incomplete orthonormal infinite sets in the complex abstract (complete) Hilbert space $\mathfrak{H}$ which is separable and infinite dimensional; let us suppose that these two orthonormal sets are mutually orthogonal and determine a complete orthonormal system in $\mathfrak{H}$; and let $(\beta_{ij}^{(\omega)})$ be a bounded normal matrix-operator with $\sum_{j=1}^{\infty} |\beta_{ij}^{(\omega)}|^2 \neq |\beta_{ii}^{(\omega)}|^2$, $i=1, 2, 3, \cdots$, in Hilbert coordinate space l_2. Then, as already shown [3], the operator $\widetilde{N}_\omega$ defined by

$$\widetilde{N}_\omega = \sum_{\nu=1}^{\infty} \lambda_\nu^{(\omega)} \varphi_\nu^{(\omega)} \otimes L_{\varphi_\nu^{(\omega)}} + c_\omega \sum_{\mu=1}^{\infty} \Psi_\mu \otimes L_{\psi_\mu^{(\omega)}} \quad \left(\Psi_\mu = \sum_{j=1}^{\infty} \beta_{\mu j}^{(\omega)} \psi_j^{(\omega)}\right)$$

is a bounded normal operator with point spectrum $\{\lambda_\nu^{(\omega)}\}$ in $\mathfrak{H}$ such that its continuous spectrum is not empty, its norm is given by $\max(\sup_\nu |\lambda_\nu^{(\omega)}|, |c_\omega| \cdot \|(\beta_{ij}^{(\omega)})\|)$, and $\varphi_\nu^{(\omega)}$ is an eigenelement of $\widetilde{N}_\omega$ corresponding to the eigenvalue $\lambda_\nu^{(\omega)}$; and if such $M, c_\omega, \{\varphi_\nu^{(\omega)}\}, \{\psi_\mu^{(\omega)}\}$, and $(\beta_{ij}^{(\omega)})$ as above are appropriately chosen, conversely any bounded normal operator with point spectrum $\{\lambda_\nu^{(\omega)}\}$ and nonempty continuous spectrum in $\mathfrak{H}$ is expressible by such a series of linear functionals $L_{\varphi_\nu^{(\omega)}}, L_{\psi_\mu^{(\omega)}}$ as above. On the assumption that M is fixed, we now denote by $\mathfrak{N}(M)$ the class of bounded normal operators $\widetilde{N}_\omega$ for all those $\{\lambda_\nu^{(\omega)}\}, c_\omega, \{\varphi_\nu^{(\omega)}\}, \{\psi_\mu^{(\omega)}\}$, and $(\beta_{ij}^{(\omega)})$ which satisfy the above conditions respectively. Moreover, for any $\widetilde{N} \in \mathfrak{N}(M)$ we denote by $\Delta(\widetilde{N})$ the continuous spectrum of $\widetilde{N}$, by $\Delta^+(\widetilde{N})$ the set of all those accumulation points of the point spectrum $\{\lambda_\nu\}$ of $\widetilde{N}$ which do not belong to $\{\lambda_\nu\}$ itself, by $\Delta^-(\widetilde{N})$ the set $\Delta(\widetilde{N}) - \Delta^+(\widetilde{N})$, and by $\{K(\zeta)\}$ the complex spectral family of $\widetilde{N}$. Then, as already pointed out in one of the preceding papers [2], $\widetilde{N}[I - K(\Delta^-(\widetilde{N}))]$ is a bounded normal operator whose point spectrum and continuous spectrum are given by $\{\lambda_\nu\}$ and $\Delta^+(\widetilde{N})$ respectively.

We shall call any $\tilde{N}[I-K(\Delta^{-}(\tilde{N}))]$ "a characterized operator of $\tilde{N}$" and shall denote by $\mathfrak{N}(M)$ the class of characterized (bounded) normal operators $\tilde{N}[I-K(\Delta^{-}(\tilde{N}))]$ for all $\tilde{N}\in\tilde{\mathfrak{N}}(M)$. Moreover $\mathfrak{N}(M)$ will be called "the characterized normal operator-class for M". By these definitions, the continuous spectrum of any bounded normal operator $N\in\mathfrak{N}(M)$ consists only of all those accumulation points of its point spectrum which do not belong to (the point spectrum) itself and the inequality $||N||\leqq M$ holds. Now let N_ω be an arbitrary operator in $\mathfrak{N}(M)$; let

$$P_\omega(\lambda)=\sum_{\alpha=1}^{m_\omega}((\lambda I-N_\omega)^{-\alpha}f_{\omega\alpha},\bar{f}_{\omega\alpha})\quad (f_{\omega\alpha}=\sum_{\nu=1}^{\infty}\sqrt{c_{\omega\alpha}^{(\nu)}}\,\varphi_\nu^{(\omega)},\bar{f}_{\omega\alpha}=\sum_{\nu=1}^{\infty}\sqrt{\bar{c}_{\omega\alpha}^{(\nu)}}\,\varphi_\nu^{(\omega)}),$$

where $1\leqq m_\omega<\infty$, $(\sqrt{c_{\omega\alpha}^{(\nu)}}\,\varphi_\nu^{(\omega)},\sqrt{\bar{c}_{\omega\alpha}^{(\nu)}}\,\varphi_\nu^{(\omega)})=c_{\omega\alpha}^{(\nu)}$, $\sum_{\nu=1}^{\infty}|c_{\omega\alpha}^{(\nu)}|<\infty$ $(\alpha=1,\cdots,m_\omega)$, and $\varphi_\nu^{(\omega)}$ denotes a normalized eigenelement of N_ω corresponding to the eigenvalue $\lambda_\nu^{(\omega)}$; let

$$Q_\omega(\lambda)=\sum_{\alpha=1}^{m_\omega}\int_{\Delta(N_\omega)}(\lambda-\zeta)^{-\alpha}d(K_\omega(\zeta)g_{\omega\alpha},h_{\omega\alpha}),$$

where $\Delta(N_\omega)$ and $\{K_\omega(\zeta)\}$ denote the continuous spectrum and the complex spectral family of N_ω respectively and both $g_{\omega\alpha}$ and $h_{\omega\alpha}$ are elements in the subspace orthogonal to the subspace determined by $\{\varphi_\nu^{(\omega)}\}_{\nu=1,2,3,\dots}$; and let $R_\omega(\lambda)$ be an arbitrary integral function. Then the function $S_\omega(\lambda)$ defined by $S_\omega(\lambda)=R_\omega(\lambda)+P_\omega(\lambda)+Q_\omega(\lambda)$ is a function possessing the same property as that on the singularities of the function $S(\lambda)$ stated in Theorem 1 [1]. Namely $P_\omega(\lambda)$ and $Q_\omega(\lambda)$ are the first and the second principal parts of $S_\omega(\lambda)$ respectively. In addition, as can be easily verified, $P_\omega(\lambda)$ and $Q_\omega(\lambda)$ are rewritten as follows:

$$P_\omega(\lambda)=\sum_{\alpha=1}^{m_\omega}\sum_{\nu=1}^{\infty}\frac{c_{\omega\alpha}^{(\nu)}}{(\lambda-\lambda_\nu^{(\omega)})^\alpha},$$

$$Q_\omega(\lambda)=\sum_{\alpha=1}^{m_\omega}((\lambda I-N_\omega)^{-\alpha}g_{\omega\alpha},h_{\omega\alpha}).$$

Since, by hypotheses, the orthonormal set $\{\varphi_\nu^{(\omega)}\}$ is incomplete, $Q_\omega(\lambda)$ never vanishes and hence the (linear or planar) measure of the continuous spectrum $\Delta(N_\omega)$ consisting only of all those accumulation points of $\{\lambda_\nu^{(\omega)}\}$ which do not belong to $\{\lambda_\nu^{(\omega)}\}$ itself is never zero. If, contrary to it, the measure of $\Delta(N_\omega)$ were zero, then $Q_\omega(\lambda)$ would vanish and the orthonormal set $\{\varphi_\nu^{(\omega)}\}$ would become complete. Throughout the present paper we shall denote by $\mathfrak{F}(M)$ the family of functions $S_\omega(\lambda)=R_\omega(\lambda)+P_\omega(\lambda)+Q_\omega(\lambda)$ for all integral functions $R_\omega(\lambda)$ and all pairs of functions $P_\omega(\lambda), Q_\omega(\lambda)$ associated with characterized (bounded) normal operators N_ω belonging to the class $\mathfrak{N}(M)$ and shall call $\mathfrak{F}(M)$ "the characterized function-family for M".

Theorem 25. Let $S_\omega(\lambda)$ and $S_\theta(\lambda)$ be arbitrary functions belonging to the characterized function-family $\mathfrak{F}(M)$ for arbitrarily prescribed positive number M; let $P_\omega(\lambda)$ and $Q_\omega(\lambda)$ be the first and the second principal parts of $S_\omega(\lambda)$ respectively; let $P_\theta(\lambda)$ and $Q_\theta(\lambda)$ be the first and the second principal parts of $S_\theta(\lambda)$ respectively; and let Γ be a rectifiable close Jordan curve, positively oriented, containing wholly the disc $|\lambda|\leqq M$ within itself. Then

$$\int_\Gamma P_\omega(\lambda)P_\theta^{(k)}(\lambda)d\lambda=\int_\Gamma P_\omega(\lambda)Q_\theta^{(k)}(\lambda)d\lambda=0 \quad (k=0,1,2,\cdots).$$

Proof. By hypotheses, there exist suitable characterized (bounded) normal operators $N_\omega, N_\theta\in\mathfrak{N}(M)$ corresponding to the two pairs of functions $(P_\omega(\lambda), Q_\omega(\lambda))$ and $(P_\theta(\lambda), Q_\theta(\lambda))$ respectively. Moreover it is clear by hypotheses that Γ contains wholly the spectra of N_ω and N_θ in the interior of itself. As can be verified immediately from the method of the proof of Theorem 2 [1], we have therefore

$$\frac{1}{2\pi i}\int_\Gamma P_\omega(\lambda)(\lambda I-N_\theta)^{-\alpha}d\lambda=0 \quad (i=\sqrt{-1})$$

for any positive integer α. On the other hand, $P_\theta(\lambda)$ is expressed in the form

$$P_\theta(\lambda)=\sum_{\alpha=1}^{m_\theta}((\lambda I-N_\theta)^{-\alpha}f_{\theta\alpha}, \bar{f}_{\theta\alpha}) \quad (\lambda\in\Gamma),$$

where $1\leqq m_\theta<\infty$ and the elements $f_{\theta\alpha}$ and $\bar{f}_{\theta\alpha}$ are appropriately chosen elements in the subspace determined by an orthonormal set of eigenelements of N_θ corresponding to all the eigenvalues. By means of these results and the relations

$$\frac{d^p}{d\lambda^p}(\lambda I-N_\theta)^{-1}=(-1)^p p!(\lambda I-N_\theta)^{-(p+1)} \quad (p=1,2,3,\cdots),$$

we obtain

$$\int_\Gamma P_\omega(\lambda)P_\theta^{(k)}(\lambda)d\lambda=0 \quad (k=0,1,2,\cdots).$$

Since, moreover, $Q_\theta(\lambda)$ is given by

$$Q_\theta(\lambda)=\sum_{\alpha=1}^{m_\theta}((\lambda I-N_\theta)^{-\alpha}g_{\theta\alpha}, h_{\theta\alpha}),$$

where both $g_{\theta\alpha}$ and $h_{\theta\alpha}$ are suitable elements in the subspace orthogonal to the subspace determined by an orthonormal set of eigenelements of N_θ corresponding to all the eigenvalues, the relations

$$\int_\Gamma P_\omega(\lambda)Q_\theta^{(k)}(\lambda)d\lambda=0 \quad (k=0,1,2,\cdots)$$

are established in the same manner as above.

Theorem 26. Let $Q_\omega(\lambda)$, $P_\theta(\lambda)$, $Q_\theta(\lambda)$, and Γ be the same notations as before. Then

$$\int_\Gamma Q_\omega(\lambda)P_\theta^{(k)}(\lambda)d\lambda=\int_\Gamma Q_\omega(\lambda)Q_\theta^{(k)}(\lambda)d\lambda=0 \quad (k=0,1,2,\cdots).$$

Proof. Let $R_\omega(\lambda)$ denote the ordinary part of $S_\omega(\lambda)$. Then, by the definition concerning the ordinary part, $R_\omega(\lambda)$ is regular on the domain $\{\lambda: |\lambda|<\infty\}$. Hence, denoting by $\{K_\theta(\zeta)\}$ the complex spectral family of N_θ, it is found that

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma R_\omega(\lambda)(\lambda I-N_\theta)^{-(k+1)}d\lambda &= \frac{1}{2\pi i}\int_\Gamma R_\omega(\lambda)\int_{|\zeta|\leqq M}(\lambda-\zeta)^{-(k+1)}dK_\theta(\zeta)d\lambda \\ &= \int_{|\zeta|\leqq M}\left\{\frac{1}{2\pi i}\int_\Gamma R_\omega(\lambda)(\lambda-\zeta)^{-(k+1)}d\lambda\right\}dK_\theta(\zeta) \\ &= \int_{|\zeta|\leqq M}\frac{R_\omega^{(k)}(\zeta)}{k!}dK_\theta(\zeta) \\ &= \frac{R_\omega^{(k)}(N_\theta)}{k!}\quad(k=0,1,2,\cdots;\ 0!=1).\end{aligned}$$

On the other hand, by reference to Theorem 3 [1], we have

$$(24)\qquad \frac{1}{2\pi i}\int_\Gamma S_\omega(\lambda)(\lambda I-N_\theta)^{-(k+1)}d\lambda=\frac{R_\omega^{(k)}(N_\theta)}{k!}\quad(k=0,1,2,\cdots).$$

By making use of the relation $\int_\Gamma P_\omega(\lambda)(\lambda I-N_\theta)^{-(k+1)}d\lambda=\mathbf{0}$ and the just established results, we have

$$\frac{1}{2\pi i}\int_\Gamma Q_\omega(\lambda)(\lambda I-N_\theta)^{-(k+1)}d\lambda=\mathbf{0}\quad(k=0,1,2,\cdots).$$

In consequence, the same reasoning as that used to prove Theorem 25 leads us to the required relations in the statement of the present theorem.

Theorem 27. Let $S_\omega(\lambda), R_\omega(\lambda), P_\theta(\lambda), M$, and Γ be the same notations as before, and let the expansion of $P_\theta(\lambda)$ be given by

$$P_\theta(\lambda)=\sum_{\alpha=1}^{m_\theta}\sum_{\nu=1}^{\infty}\frac{c_{\theta\alpha}^{(\nu)}}{(\lambda-\lambda_\nu^{(\theta)})^\alpha}\quad(1\leqq m_\theta<\infty),$$

where $\sum_{\nu=1}^{\infty}|c_{\theta\alpha}^{(\nu)}|<\infty$ for $\alpha=1,\cdots,m_\theta$ and $\sup_\nu|\lambda_\nu^{(\theta)}|\leqq M$. Then

$$\frac{1}{2\pi i}\int_\Gamma S_\omega(\lambda)P_\theta(\lambda)d\lambda=\sum_{\alpha=1}^{m_\theta}\frac{1}{(\alpha-1)!}\sum_{\nu=1}^{\infty}c_{\theta\alpha}^{(\nu)}R_\omega^{(\alpha-1)}(\lambda_\nu^{(\theta)}),$$

where the series on the right-hand side converges absolutely.

Proof. Since, by hypotheses, the set $\{\lambda_\nu^{(\theta)}\}_{\nu=1,2,3,\ldots}$ is the point spectrum of the characterized normal operator $N_\theta\in\mathfrak{N}(M)$ corresponding to the pair of $P_\theta(\lambda)$ and $Q_\theta(\lambda)$,

$$(25)\qquad P_\theta(\lambda)=\sum_{\alpha=1}^{m_\theta}((\lambda I-N_\theta)^{-\alpha}f_{\theta\alpha},\bar{f}_{\theta\alpha})\quad(N_\theta\varphi_\nu^{(\theta)}=\lambda_\nu^{(\theta)}\varphi_\nu^{(\theta)}),$$

where $f_{\theta\alpha}=\sum_{\nu=1}^{\infty}\sqrt{c_{\theta\alpha}^{(\nu)}}\varphi_\nu^{(\theta)}$, $\bar{f}_{\theta\alpha}=\sum_{\nu=1}^{\infty}\sqrt{\bar{c}_{\theta\alpha}^{(\nu)}}\varphi_\nu^{(\theta)}$, $(\sqrt{c_{\theta\alpha}^{(\nu)}}\varphi_\nu^{(\theta)},\ \sqrt{\bar{c}_{\theta\alpha}^{(\nu)}}\varphi_\nu^{(\theta)})=c_{\theta\alpha}^{(\nu)}$, and $\sum_{\nu=1}^{\infty}|c_{\theta\alpha}^{(\nu)}|<\infty$ $(\alpha=1,\cdots,m_\theta)$. Since, on the other hand, $R_\omega(\lambda)$ is expressible in the form $R_\omega(\lambda)=\sum_{n\geqq 0}a_\omega^{(n)}\lambda^n$ $(|\lambda|<\infty)$,

$$\begin{aligned}(R_\omega^{(k)}(N_\theta)f_{\theta\alpha},\bar{f}_{\theta\alpha})&=(\sum_{n\geqq k}n(n-1)\cdots(n-k+1)a_\omega^{(n)}N_\theta^{n-k}f_{\theta\alpha},\bar{f}_{\theta\alpha})\\ &=\sum_{r\geqq 0}\frac{(k+r)!}{r!}a_\omega^{(k+r)}\sum_{\nu=1}^{\infty}c_{\theta\alpha}^{(\nu)}(\lambda_\nu^{(\theta)})^r,\end{aligned}$$

for which

$$\sum_{r\geqq 0}\frac{(k+r)!}{r!}|a_\omega^{(k+r)}|\sum_{\nu=1}^{\infty}|c_{\theta\alpha}^{(\nu)}||(\lambda_\nu^{(\theta)})^r|\leqq\widetilde{R}_\omega^{(k)}(M)\sum_{\nu=1}^{\infty}|c_{\theta\alpha}^{(\nu)}|<\infty$$

$$(\widetilde{R}_\omega(|\lambda|)\equiv\sum_{n\geqq 0}|a_\omega^{(n)}||\lambda|^n)$$

by virtue of the hypothesis $\sup_\nu|\lambda_\nu^{(\theta)}|\leqq M$. Hence it is easily found that

$$(R_\omega^{(k)}(N_\theta)f_{\theta\alpha},\bar{f}_{\theta\alpha})=\sum_{\nu=1}^{\infty}c_{\theta\alpha}^{(\nu)}R_\omega^{(k)}(\lambda_\nu^{(\theta)}).$$

By applying this final result and the relations (24) and (25), we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma S_\omega(\lambda)P_\theta(\lambda)d\lambda&=\frac{1}{2\pi i}\int_\Gamma S_\omega(\lambda)\sum_{\alpha=1}^{m_\theta}((\lambda I-N_\theta)^{-\alpha}f_{\theta\alpha},\bar{f}_{\theta\alpha})d\lambda\\&=\sum_{\alpha=1}^{m_\theta}\frac{1}{(\alpha-1)!}(R_\omega^{(\alpha-1)}(N_\theta)f_{\theta\alpha},\bar{f}_{\theta\alpha})\\&=\sum_{\alpha=1}^{m_\theta}\frac{1}{(\alpha-1)!}\sum_{\nu=1}^{\infty}c_{\theta\alpha}^{(\nu)}R_\omega^{(\alpha-1)}(\lambda_\nu^{(\theta)}).\end{aligned}$$

Thus the present theorem has been proved.

Corollary 4. Let Δ be a Lebesgue μ-measurable set of finite or infinite measure in real m-dimensional Euclidean space; let $L_2(\Delta,\mu)$ be the Lebesgue (square integrable) function-space; let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be an arbitrarily prescribed bounded infinite sequence of complex numbers (counted according to the respective multiplicities); let M be a positive constant with $\sup_\nu|\lambda_\nu|\leqq M$; let $\mathfrak{F}(M)$ be the characterized function-family for M; let Γ be a rectifiable closed Jordan curve, positively oriented, such that the disc $|\lambda|\leqq M$ lies within Γ itself; let $\{\varphi_\nu(x)\}_{\nu=1,2,3,\dots}$ be a complete orthonormal system in $L_2(\Delta,\mu)$; let N be the operator defined by $(Nf)(x)=\sum_{\nu=1}^{\infty}\lambda_\nu\int_\Delta f(y)\overline{\varphi_\nu(y)}d\mu(y)\cdot\varphi_\nu(x)$ for every $f\in L_2(\Delta,\mu)$; and let $f(x,\lambda)$ be the solution of the equation $\lambda f(x)-(Nf)(x)=g(x)$ ($g\in L_2(\Delta,\mu)$, $\lambda\in\Gamma$). Then, for the first and the second principal parts $P_\omega(\lambda)$, $Q_\omega(\lambda)$ and the ordinary part $R_\omega(\lambda)$ of any $S_\omega(\lambda)\in\mathfrak{F}(M)$ and for almost every $x\in\Delta$,

$$\int_\Gamma P_\omega(\lambda)f(x,\lambda)d\lambda=\int_\Gamma Q_\omega(\lambda)f(x,\lambda)d\lambda=0$$

and

$$\frac{1}{2\pi i}\int_\Gamma S_\omega(\lambda)f(x,\lambda)d\lambda=\sum_{\nu=1}^{\infty}R_\omega(\lambda_\nu)\int_\Delta g(y)\overline{\varphi_\nu(y)}d\mu(y)\cdot\varphi_\nu(x),$$

where the series on the right is a function in $L_2(\Delta,\mu)$.

Proof. By hypotheses, there is no difficulty in showing that N is a bounded normal operator with point spectrum $\{\lambda_\nu\}$ in $L_2(\Delta,\mu)$ and that

$$f(x,\lambda)=\sum_{\nu=1}^{\infty}\frac{1}{\lambda-\lambda_\nu}\int_\Delta g(y)\overline{\varphi_\nu(y)}d\mu(y)\cdot\varphi_\nu(x)\quad(\lambda\in\Gamma)$$

in the sense of convergence in mean on Δ. If, for the sake of simplicity, $f(x, \lambda)$ is denoted by f_λ, then, for every non-null element $h \in L_2(\Delta, \mu)$, the function $((\lambda I-N)^{-1}g, h)=(f_\lambda, h)$ of λ is regarded as the first principal part of a special function whose ordinary part and second principal part both vanish. On the other hand, Theorems 25, 26, and 27 hold also in the special case where $R_\theta(\lambda)=Q_\theta(\lambda)=0$ or $R_\theta(\lambda)=P_\theta(\lambda)=0$, as will be seen from the methods of the proofs of those theorems. Accordingly both $\int_\Gamma P_\omega(\lambda)f_\lambda d\lambda$ and $\int_\Gamma Q_\omega(\lambda)f_\lambda d\lambda$ are orthogonal to every $h \in L_2(\Delta, \mu)$, and so also is $\frac{1}{2\pi i}\int_\Gamma S_\omega(\lambda)f_\lambda d\lambda-\sum_{\nu=1}^{\infty}R_\omega(\lambda_\nu)(g, \varphi_\nu)\varphi_\nu$ by virtue of the relation

$$\frac{1}{2\pi i}\int_\Gamma S_\omega(\lambda)(f_\lambda, h)d\lambda=\sum_{\nu=1}^{\infty}R_\omega(\lambda_\nu)(g, \varphi_\nu)(\varphi_\nu, h).$$

These results permit us to conclude that the relations in the statement of the present corollary are valid. Moreover, from Parseval's identity and the boundedness of the set $\{R_\omega(\lambda_\nu)\}_{\nu=1,2,3,\dots}$, it is obvious that $\sum_{\nu=1}^{\infty}R_\omega(\lambda_\nu)(g, \varphi_\nu)\varphi_\nu$ belongs to $L_2(\Delta, \mu)$, as we wished to prove.

References

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