109. *Extension of a Certain C* Algebra*

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§1. **Introduction.** All field operators such as creation operator, annihilation operator and the values of field function obtained by the cut off process are unbounded operators. Then C^* algebra consisted of all bounded operators is not necessarily sufficient as the set of observables. For the investigation of the various characters of field operators, and for using the results of many researches with respect to C^* algebra effectively, its suitable extension is needed. The weak topology used by R. Haag and D. Kastler in [1] becomes to one powerful tool to extend C^* algebra by its suitable use. In this paper, we don't device the suitable use of the weak topology, but we will consider the meaning of the domain of selfadjoint unbounded operators, and show the limitation of the most usual weak extension of this C^* algebra from deeper view point than [2].

On the other hand, another extension using E. R. Integral has been shown in [2]. The generalized mathematical expectation by using A-integral (equivalent to E. R. Integral) has been already defined by Kolmogorov in the most primitive form [7]. But quantized solution of the Klein Golden equation, etc. cannot be treated by this method. Here, we will show the more extended concept of the observables by using n dimensional E. R. Integral and give the definition of the observables containing the above solutions, etc.

§2. **Unbounded operators produced by Weak extension.** Here, we only consider the linear operator T densely defined in $\mathfrak{H}$. The definition of unbounded operator is the following usual one.

Definition 1. If there is not a fixed bounded number C such that $||T\varphi||\leqq C||\varphi||$ for all φ contained in the domain, we call this operator T an unbounded linear operator.

From the definition of the usual weak topology, we classify the set of unbounded linear operators to the following two types:

(1) the domain of T is $\mathfrak{H}$,

(2) the domain of T is purely contained in $\mathfrak{H}$.

Let's show the examples belonging to each class.

Example 1. Let $\{e_n\}$ be a base in $\mathfrak{H}$, let D be the set of the finite linear aggregate of e_n and let T be a linear operator defined in the set D with the property $Te_n=ne_n$.

At the first step, using this D, let's classify $\mathfrak{H}$ and construct the space of the classes $\mathfrak{H}/D$.

Next, after this classification let's choose the base $\{x_j\}$ of $\mathfrak{H}/D$. From Zermero's axiom, it is obvious that this choice is possible. From each class x_j of this base, let's select an element x_j in $\mathfrak{H}$ and define the extended operator $\widetilde{T}$ related to T with the properties $\widetilde{T}x_j=0$ for the above x_j. The domain of this extended unbounded operator $\widetilde{T}$ is $\mathfrak{H}$.

Example 2. The above T (in Example 1) is the operator in the case (2).

But the more natural example can be constructed.

Example 3. The creation operator defined in the possible domain is the example of the case (2).

Let's compare the above three kind of Examples. All these operators are not continuous, but these examples have the differences whether it is closed or not.

Now let's show the definition of the closed unbounded operator.

Since unbounded operator T is densely defined, we can choose the base $\{e_n\}$ of the separable Hilbert space $\mathfrak{H}$ contained in the domain of T. By using the Zermero's axiom, this choice is also possible for the non separable Hilbert space $\mathfrak{H}$.

Definition 2. If the set of the possible pair (x, Tx) for an unbounded operator T is closed in $\mathfrak{H}\times\mathfrak{H}$, we call this operator T closed unbounded operator. We call this set $\{(x, Tx)\}$ the graph of T.

Definition 3. If unbounded operator T is not closed, we call this T non-closed unbounded operator.

Example 1 shows non-closed unbounded operator such that $D_{\widetilde{T},\{e_n\}}\subset D_{\widetilde{T}}=\mathfrak{H}$. Here $D_{\widetilde{T},\{e_n\}}$ is the set (in $\mathfrak{H}$) $\{x; x=\Sigma_n C_n e_n\in\mathfrak{H}, \Sigma_n C_n\widetilde{T}e_n\in\mathfrak{H}\}$ and $D_{\widetilde{T}}$ is the domain of T. Example 2 shows an unbounded operator such that $D_{T,\{e_n\}}\supset D_T\supset D$, where D is the finite linear aggregate of e_n used in Example 1. Example 3 shows a closed unbounded operator. For each non-closed unbounded operator T, there exists at least one closed unbounded operator $T^{(N)}$ which is equal to T in D (used in Example 1). Here, this D is constructed from some base $\{e_n\}$ of $\mathfrak{H}$ in the domain of T. In Example 2 $T^{(N)}$ is obtained compatibly (to original T). The closed unbounded operator $T^{(N)}$ which is obtained from T compatibly (to the original T) is not always one. Because it depends upon the method of the closed extension. By the same method as Example 1, we can obtain the closed linear unbounded operator whose restriction is the operator T in Example 3. From Toeplitz's theorem, it is obvious that the closed linear unbounded operator defined in all $\mathfrak{H}$ can not be obtained [4]. Physical operator such as creation or annihilation operator etc. is self adjoint. It can be represented by the spectral decomposed form. Let's denote it by

$T=\int \lambda dE(\lambda)$. Since T is not a bounded operator, the spectral measure $dE(\lambda)$ is also distributed in the complement of any bounded interval. The element of the Hilbert space $\mathfrak{H}$ is represented by $\varphi=\int \varphi(\lambda)dE(\lambda)$ which has the character $\int |\varphi(\lambda)|^2 d||E(\lambda)||^2<+\infty$. This representation is a sort of generalization of the orthogonal decomposition. D_T is the set of the element $\varphi \in \mathfrak{H}$ such that $\int |\lambda\varphi(\lambda)|^2 d||E(\lambda)||^2<+\infty$.

From the above considerations we see that these physical unbounded operators are closed unbounded operators, and its domain depends on the style of improper integral.

Furthermore, we can extend the domain of the unbounded operators as follows.

Definition 4. 1) The domain of linear unbounded operator T is $D_T=\{\varphi; \varphi \in \mathfrak{H}, T\varphi \in \mathfrak{H}\} \subset \mathfrak{H}$. 2) The extended domain of linear unbounded operator T is $\tilde{D}_T=\{\varphi; \varphi\in\mathfrak{H}, T\varphi\in\mathfrak{H}$ or $||T\varphi||=+\infty$ by some meaning$\}\subset\mathfrak{H}$.

For unbounded operator T, compatible $\tilde{D}_T(\neq D_T)$ cannot be obtained in usual. For example, the domain of T in Example 1 is $\mathfrak{H}$. Even the compatible $D_{T,\{e_n\}}$ cannot be obtained.

If $\tilde{D}_T$ is purely contained in $\mathfrak{H}$, this unbounded operator T is not obviously contained in the weak extension of the set of bounded operators by means of R. Haag and D. Kastler.

Hence in [2] we show that the operator T with the following properties is not contained in the weak extension of the set of bounded operators.

The properties are

(a) the compatible definition of $\tilde{D}_T(\neq D_T)$ is possible, and

(b) $\tilde{D}_T=\mathfrak{H}$.

Even if there is only one element φ such that $||T\varphi||=\infty$, the result in [2] holds valid. Hence, at last, we must consider the possibility of the self-adjoint unbounded operator (not necessarily closed) defined in all elements of $\mathfrak{H}$ similar to Example 1.

The following lemma shows that the self-adjoint unbounded operators must be closed and these operators defined in $\mathfrak{H}$ (in the usual meaning) cannot be considered.

Lemma 1. *If T is a self-adjoint operator, and if $T\Sigma C_n e_n$ is contained in $\mathfrak{H}$, then it must be $\Sigma C_n Te_n$.*

Proof. Let's choose a base $\{e_n\}$ in $\mathfrak{H}$ and select sequence $\{C_n\}$ such that $\Sigma_n C_n e_n \in \mathfrak{H}$ and $T\Sigma_n C_n e_n\in\mathfrak{H}$. Since $\langle e_m, T\Sigma_n C_n e_n\rangle=\langle Te_m, \Sigma_n C_n e_n\rangle$ for any m, it follows that $\langle \Sigma_n C_n e_n, T\Sigma_n C_n e_n\rangle=\langle T\Sigma_n C_n e_n, \Sigma_n C_n e_n\rangle=\langle \Sigma_n C_n e_n, \Sigma_n C_n Te_n\rangle$.

For any finite sum $\Sigma_m d_m \boldsymbol{e}_m$ contained in D (Example 1), $\langle \Sigma_m d_m \boldsymbol{e}_m, T\Sigma_n C_n \boldsymbol{e}_n \rangle \equiv (1/4)\ \{(\Sigma_n (d_n + C_n)\boldsymbol{e}_n,\ T\cdot\Sigma_n (d_n + C_n)\boldsymbol{e}_n) - (\Sigma_n (d_n - C_n)\boldsymbol{e}_n,\ T\cdot\Sigma_n (d_n - C_n)\boldsymbol{e}_n) + i(\Sigma_n (d_n + iC_n)\boldsymbol{e}_n,\ T\cdot\Sigma_n (d_n + iC_n)\boldsymbol{e}_n) - i(\Sigma_n (d_n - iC_n)\boldsymbol{e}_n,\ T\cdot\Sigma_n (d_n - iC_n)\boldsymbol{e}_n)\} = (1/4)\{(\Sigma_n (d_n + C_n)\boldsymbol{e}_n, \Sigma_n (d_n + C_n)T\boldsymbol{e}_n) - (\Sigma_n (d_n - C_n)\boldsymbol{e}_n, \Sigma_n (d_n - C_n)T\boldsymbol{e}_n) + i(\Sigma_n (d_n + iC_n)\boldsymbol{e}_n,\ \Sigma_n (d_n + iC_n)T\boldsymbol{e}_n) - i(\Sigma_n (d_n - iC_n)\boldsymbol{e}_n,\ \Sigma_n (d_n - iC_n)T\boldsymbol{e}_n)\} \equiv \langle \Sigma_m d_m \boldsymbol{e}_m, \Sigma_n C_n T\boldsymbol{e}_n \rangle$.

Hence if $T\Sigma_n C_n \boldsymbol{e}_n$ is contained in $\mathfrak{H}$, then $T\Sigma_n C_n \boldsymbol{e}_n = \Sigma_n C_n T\boldsymbol{e}_n$.

The essential tool in Lemma 1 is the adjoint operation. Then the following extension of Lemma 1 is obtained.

Lemma 2. *If T has an adjoint operator T^*, if $T^*\boldsymbol{e}_n \in \mathfrak{H}$ for every n and if $T(\Sigma_m C_m \boldsymbol{e}_m)$ is contained in $\mathfrak{H}$, then it must be $\Sigma_m C_m T\boldsymbol{e}_m$.*

Proof. For $\Sigma_n C_n \boldsymbol{e}_n \in \mathfrak{H}$, $\langle \Sigma_n C_n T\boldsymbol{e}_n,\ \Sigma_n C_n \boldsymbol{e}_n \rangle = \langle \Sigma_n C_n \boldsymbol{e}_n,\ T^*\Sigma_n C_n \boldsymbol{e}_n \rangle = \langle T\Sigma_n C_n \boldsymbol{e}_n,\ \Sigma_n \boldsymbol{e}_n \rangle$ holds good.

Then $T(\Sigma_n C_n \boldsymbol{e}_n) = \Sigma_n C_n T\boldsymbol{e}_n$ by the same way as Lemma 1.

From these lemmas we see that the self-adjoint unbounded operator T is defined by $\Sigma_n C_n T\boldsymbol{e}_n$ in its domain and the self-adjoint unbounded operator T with the domain $\mathfrak{H}$ cannot be considered. Then, at last, the following theorem is obtained.

Theorem 1. *Physical unbounded operators whose adjoint can be obtained are not contained in the weak closure of the set of bounded operators.*

The compatible extended domain $\widetilde{D}_T (\neq D_T)$ can be defined naturally. By the suitable summations rule, the domain of T can be extended and the extended domain is obtained. Namely, the summations rule effect to the condition about the domain of T directly.

Hence we see that the rule of conditional convergence has special important meaning. One of them is the following E. R. Integral.

§3. **E. R. Extension.** In this paragraph let's try to represent the expectation value of field functions $\varphi(\boldsymbol{x})$ and $\varphi_V(\boldsymbol{x})$ using E. R. Integral.

$\varphi(\boldsymbol{x})$ and $\varphi_V(\boldsymbol{x})$ can be written down by the following form:

$$\varphi(\boldsymbol{x}) = (1/(2\pi)^{3/2})\left\{\int a^+(\boldsymbol{k})e^{i\boldsymbol{kx}}d\boldsymbol{k} + \int a(\boldsymbol{k})e^{-i\boldsymbol{kx}}d\boldsymbol{k}\right\}$$

and
$$\varphi_V(\boldsymbol{x}) = (1/\sqrt{V})[\Sigma_{\boldsymbol{k}=(k_1,k_2,k_3)}(a^+(\boldsymbol{k})e^{i\boldsymbol{kx}} + a(\boldsymbol{k})e^{-i\boldsymbol{kx}})].$$

From $\varphi_V(\boldsymbol{x})$, the following cut off function $\varphi_{V,M}(\boldsymbol{x})$ can be obtained;

$$\varphi_{V,M}(\boldsymbol{x}) = (1/\sqrt{V})[\Sigma_{|\boldsymbol{k}|\leqq M}(a^+(\boldsymbol{k})e^{i\boldsymbol{kx}} + a(\boldsymbol{k})e^{-i\boldsymbol{kx}})].$$

Let's represent the expectation value of $\varphi_V(\boldsymbol{x})$ by the integral of the function $f(\boldsymbol{x}, u, v)$ (for $\boldsymbol{x} \in E^4$) defined in the interval $0 < u \leqq 1$ $0 < v \leqq 1$ as follows.

$f(\boldsymbol{x}, u, v) = 2 \times f_1(\boldsymbol{x}, u)$ for $1/2 < v \leqq 1$,

$f(\boldsymbol{x}, u, v) = 2^2 \times \{f_2(\boldsymbol{x}, u) - f_1(\boldsymbol{x}, u)\}$ for $(1/2)^2 < v \leqq 1/2$,

. . . .

$f(\boldsymbol{x}, u, v) = 2^M \times \{f_M(\boldsymbol{x}, u) - f_{M-1}(\boldsymbol{x}, u)\}$ for $(1/2)^M < v \leqq (1/2)^{M-1}, \cdots$.

Here $f_M(x, u)$ is one of the function defined in [2]. $\int_0^1\int_0^1 f(x, u, v)\,dudv$ is definite for any $x \in E^4$ by improper E. R. Integral.

Now let's show the determination of the function $f_M(x, u)$ corresponding to the expectation value of the field function $\varphi_{V,M}(x)$ precisely, at once.

Let's decompose ψ_j $(j=1, 2)$ in $\psi_j=\Sigma_i C_i^{(j)}\varphi_i=\Sigma_i C_i^{(j)}\Pi_k\otimes\varphi(n_k^{(i)})$ and construct $\psi_{j,M}=\Sigma_i C_i^{(j)}\varphi_{i,M}=\Sigma_i C_i^{(j)}\Pi_{|k|\leq M}\otimes\varphi(n_k^{(i)})$ from ψ_j using a sort of cutoff.

Furthermore, decompose $\varphi_{V,M}(x)\psi_{2,M}$ in $\Sigma_i C_{i,M}^{(A)}(x)\varphi_{iM}$.

Then, $(\psi_{1,M}, \varphi_{V,M}(x)\psi_{2,M})$ can be represented by the infinite sum $\Sigma_i C_i^{(1)}\overline{C}_{i,M}^{(A)}(x)$.

$\varphi_{V,M}(x)$ has the following properties:

(1) For any $\psi_2\in\mathfrak{H}$, $C_{i,M}^{(A)}(x)$ are finite and fixed for all x and i.

(2) $\Sigma_i C_i^{(1)}\overline{C}_{i,M}^{(A)}(x)$ determined by the following integral is definite. (Finiteness is omitted.)

Now, let's determine its integrand.

At the first step, by ordering the positive integer valued functions $n_{k,M}$ defined in K_M, we obtain the sequence $n_{k,M}^{(1)}\, n_{k,M}^{(2)}\cdots$ with the property $\Sigma_{K_M} n_{k,M}^{(l)}\leqq\Sigma_{K_M} n_{k,M}^{(s)}$ for $l<s$, where K_M is the bounded set of the momentum k such that $|k|\leqq M$.

Next, define the following function $f_M(x, u)$,

$$f_M(x, u)=2\times C_{(1),M}(x) \quad \text{for } 1/2<u\leqq 1$$
$$f_M(x, u)=2^2\times C_{(2),M}(x) \quad \text{for } (1/2)^2<u\leqq 1/2$$
$$\cdots\cdots\cdots\cdots$$
$$f_M(x, u)=2^s\times C_{(s),M}(x) \quad \text{for } (1/2)^s<u\leqq(1/2)^{s-1},\cdots,$$

where $C_{(s),M}(x)=C_s^{(1)}\overline{C}_{s,M}^{(A)}(x)$. Its integrand is this $f_M(x, u)$ and $\Sigma_i C_i^{(1)} C_{i,M}^{(A)}(x)=\int_0^1 f_M(x, u)du$ by improper E. R. Integral. The expectation values of $\varphi(x)$ can be represented by the integral of the function $f(x, u, v, w)$ (for $x\in E^4$) defined in the interval $0<u\leqq 1$, $0<v\leqq 1$, $0<w\leqq 1$ as follows.

$$f(x, u, v, w)=2\times f_1(x, u, v) \quad \text{for } 1/2<w\leqq 1,$$
$$f(x, u, v, w)=2^2\times\{f_2(x, u, v)-f_1(x, u, v)\} \quad \text{for } (1/2)^2<w\leqq 1/2$$
$$\cdots\cdots\cdots\cdots$$
$$f(x, u, v, w)=2^p\times\{f_p(x, u, v)-f_{p-1}(x, u, v)\}$$
$$\text{for } (1/2)^p<w\leqq(1/2)^{p-1},\cdots.$$

Here $f_p(x, u, v)$ is the above function corresponding to $\varphi_{V(p^3)}(x)$. ($V(p^3)$ is a cube with volume p^3 and with center 0.) Its integral $\int_0^1\int_0^1\int_0^1 f(x, u, v, w)\,du\,dv\,dw$ is definite for any x by improper E. R. Integral.

Using the above arguments, we can extend expectation values. If a function with respect to $\boldsymbol{x}$ can construct the limit expectation value with respect to n parameters which is definite for any elements in $\mathfrak{H}$ and $\boldsymbol{x}$ in E^4, we say that it is the extended observables of order n.

Namely $\varphi_V(\boldsymbol{x})$ is the extended observable of order two and $\varphi(\boldsymbol{x})$ is the extended observable of order three. In this definition we can use the advantage of E. R. Integral with respect to the demension of the domain of integrand. But this representation depends upon the sequence of V and M. We cannot avoid this [6].

References

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