105. A Remark on a Construction of Finite Factors. II

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1. In the previous paper [2], we proved that the crossed product $G\otimes\mathcal{A}$ of a finite von Neumann algebra $\mathcal{A}$ by a group G of outer automorphisms has the property Q (Definition 2, in the below), only if G is amenable, and that the factor constructed by an enumerable ergodic m-group G on a measure space by the method due to Murray and von Neumann [4] is a continuous hyperfinite factor only if G is amenable.

In the present note, we shall show that the crossed product of a finite von Neumann algebra with the property Q by an amenable group G of outer automorphisms has the property P in the sense of J. T. Schwartz [5] (Definition 1, in the below), and that the factor constructed by an enumerable ergodic amenable m-group G and a measure space by the method due to Murray and von Neumann has the property P.

We shall use the terminology due to Dixmier [3] and the previous note [2] without further explanations.

2. In the first place, we state some properties of the operator Banach mean defined in [1]. Let G be a discrete group and $L^\infty(G)$ the algebra of all bounded complex-valued functions on G. We shall denote a Banach mean on $L^\infty(G)$ by $\int_G x(g)dg$. A group with a Banach mean will be called *amenable*. Let $\{T_g;\, g\in G\}$ be an operator family which is uniformly bounded on a Hilbert space. If G is amenable, then

$$[x|y]=\int_G (T_g x|y)dg$$

is a bounded bilinear form on the Hilbert space. Hence, there exists a unique bounded operator T such that $[x|y]=(Tx|y)$. Then we shall call T the *operator Banach mean* on G and write it by

$$T=\int_G T_g\, dg.$$

It is proved in [1] that the operator Banach mean satisfies the following properties:

a) $\int[\alpha T_g+\beta S_g]dg=\alpha\int T_g dg+\beta\int S_g dg,$

b) $\int T_g dg \geqq 0$ if $T_g \geqq 0$ for all g,

c) $\int T_g^* dg = \left[\int T_g dg\right]^*$,

d) $\int T_{gh} dg = \int T_g dg$, for any $h \in G$,

e) $\int I\, dg = I$,

f) $\int ST_g\, dg = S\int T_g\, dg$,

g) $\int T_g S\, dg = \left[\int T_g\, dg\right] S$,

where S is a bounded operator on the Hilbert space;

h) Let $\mathcal{K}$ be a weakly closed convex set of operators on the Hilbert space, and suppose $T_g \in \mathcal{K}$ for all $g \in G$, then

$$\int T_g\, dg \in \mathcal{K},$$

i) Let T be an operator, and suppose that T commutes with T_g for all $g \in G$. Then $\int T_g\, dg$ commutes with T.

The following definition is due to Schwartz [5]:

DEFINITION 1. A von Neumann algebra $\mathcal{A}$ has the *property P* if for each linear operator T in the Hilbert space the weakly closed convex hull $\mathcal{K}_T$ of the set $\{UTU^*; U \in \mathcal{A}, U \text{ unitary}\}$ has a non-void intersection with $\mathcal{A}'$.

The following definition is a purely algebraical version of Definition 1 which is introduced in [1]:

DEFINITION 2. A von Neumann algebra $\mathcal{A}$ has the *property Q* if there exists an amenable group $\mathcal{G}$ of unitary operators which generates $\mathcal{A}$. In this case, $\mathcal{G}$ will be called an *amenable generator* of $\mathcal{A}$.

THEOREM 1. *Let $\mathcal{A}$ be a von Neumann algebra with a finite faithful normal trace φ and G a group of outer automorphisms of $\mathcal{A}$ which satisfies the following property:*

$$\varphi(A^g) = \varphi(A) \qquad \text{for } A \in \mathcal{A} \text{ and } g \in G.$$

If $\mathcal{A}$ has the property Q and G is amenable, then the crossed product $G \otimes \mathcal{A}$ of $\mathcal{A}$ by G with respect to φ has the property P.

The proof of Theorem 1 has some analogy with the technique of Lemma 3 of Schwartz [6]. Let us identify $\mathcal{A}$ with $1 \otimes \mathcal{A}$. Let $\mathcal{G}$ be an amenable generator of $\mathcal{A}$ and

$$\Phi_0(T) = \int_{\mathcal{G}} UTU^* dU \quad \text{for any } T \in \mathcal{L}(G \otimes \mathcal{H}),$$

where $\mathcal{H}$ is the representation space of $\mathcal{A}$ with respect to φ. Then $\Phi_0(T) \in \mathcal{A}'$ by d), f), and g). On the other hand, $U_g A U_g^{-1} = A^{g^{-1}}$, where U_g is defined by

$$U_g(\Sigma_h h\otimes A_h)=\Sigma_h gh\otimes A_h^{g^{-1}},$$

and so $U_g\mathcal{A}U_g^{-1}=\mathcal{A}$. Hence $U_g\mathcal{A}'U_g^{-1}=\mathcal{A}'$ for any $g\in G$. If we put

$$\Phi(T)=\int_G U_g\Phi_0(T)U_g^{-1}\,dg,$$

then $U_g\Phi_0(T)U_g^{-1}\in U_g\mathcal{A}'U_g^{-1}=\mathcal{A}'$. Thus $\Phi(T)\in\mathcal{A}'$. We have

$$U_h\Phi(T)U_h^{-1}=\int_G U_hU_g\Phi_0(T)U_g^{-1}U_h^{-1}dg=\int_G U_{hg}\Phi_0(T)U_{hg}^{-1}dg$$

$$=\int_G U_g\Phi_0(T)U_g^{-1}dg=\Phi(T).$$

Therefore $\Phi(T)$ commutes with $\mathcal{A}$ and $\{U_h;h\in G\}$. Since $G\otimes\mathcal{A}$ is generated by $\mathcal{A}$ and $\{U_h;h\in G\}$, we have $\Phi(T)\in(G\otimes\mathcal{A})'$. Every $U\in\mathcal{G}$ and U_g $(g\in G)$ are unitary operators of $G\otimes\mathcal{A}$, and $\Phi_0(T)\in\mathcal{K}_T$ so that $\Phi(T)\in\mathcal{K}_{\Phi_0(T)}\subset\mathcal{K}_T$. Hence $\Phi(T)\in\mathcal{K}_T\cap(G\otimes\mathcal{A})'$. Therefore $G\otimes\mathcal{A}$ has the property P.

THEOREM 2. *Let G be an enumerable ergodic m-group in a measure space (S,m) with $m(S)=1$. If G is amenable, then the factor constructed by G and (S,m) by the method of Murray and von Neumann has the property P.*

Since Turumaru pointed out in [7] that the factor constructed by G and (S,m) by the method due to Murray and von Neumann is nothing but the crossed product of a certain abelian von Neumann algebra by a group of outer automorphisms which is isomorphic to G, Theorem 2 follows from Theorem 1.

In this paper, we obtained a sufficient condition for the crossed product to have the property P. We shall discuss elsewhere a sufficient condition for the crossed product to have the property Q.

References

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