104. A Remark on a Construction of Finite Factors. I

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1. In the previous paper [1], we have introduced in von Neumann algebras the following algebraical property:

DEFINITION. A von Neumann algebra $\mathcal{A}$ has the *property Q*, if there exists an amenable group $\mathcal{G}$ of unitary operators of $\mathcal{A}$, which generates $\mathcal{A}$. In this case, the group $\mathcal{G}$ will be called an *amenable generator* of $\mathcal{A}$.

In the present note, we shall treat a relation between the property Q and the crossed product of von Neumann algebras introduced by Turumaru [8]. Actually, we shall show that G is amenable whenever $G\otimes\mathcal{A}$ has the property Q in Theorem 1. Consequently, we can reproduce in Theorem 2 that the crossed product of the hyperfinite continuous factor by a certain group of outer automorphisms is not hyperfinite, cf. [4]. Furthermore, we shall see in Theorem 3 that the so-called measure theoretic construction of Murray and von Neumann [5] produces the hyperfinite factor only if the given ergodic group of measure-preserving transformations is amenable.

In what follows, we shall use the terminologies and the notations of Dixmier's text book [2] without further explanations.

2. Since the discussion to follow makes heavy use of the concept of the crossed product of von Neumann algebras developed by [3], [7] and [8], we review this matter briefly.

We shall denote a function on an enumerable group G of outer automorphisms of a von Neumann algebra $\mathcal{A}$ by $\Sigma_g g\otimes A_g$ where $A_g\in\mathcal{A}$ is the value of the function at $g\in G$. Let $\mathcal{D}$ be the set of all functions such that $A_g=0$ up to a finite subset of G. Besides the usual addition and the scalar multiplication, $\mathcal{D}$ becomes a *-algebra by the following operations:

$$(\Sigma_g g\otimes A_g)(\Sigma_h h\otimes B_h)=\Sigma_{g,h} gh\otimes A_g B_h^{g^{-1}}$$

and

$$(\Sigma_g g\otimes A_g)^*=\Sigma_g g^{-1}\otimes A_g^{g*}.$$

Let φ_0 be a faithful normal trace of $\mathcal{A}$ which is invariant under G, then we shall introduce the trace φ in $\mathcal{D}$ by

$$\varphi(g\otimes A_g)=\begin{cases}\varphi_0(A_g) & \text{for } g=1\\ 0 & \text{for } g\neq 1\end{cases}$$

and

$$\varphi(\Sigma_g g\otimes A_g)=\Sigma_g \varphi(g\otimes A_g).$$

Let $\mathcal{H}$ be the representation space of $\mathcal{A}$ by φ_0, then $G\otimes\mathcal{H}$, in the sense of Umegaki [8], is the representation space of $\mathcal{D}$ by φ, and $\mathcal{D}$ is represented faithfully on $G\otimes\mathcal{H}$ since φ is faithful on $\mathcal{D}$. If we define the operator $1\otimes A$ on $G\otimes\mathcal{H}$ for $A\in\mathcal{A}$ by

$$(1\otimes A)(\Sigma_g\, g\otimes A_g)=\Sigma_g\, g\otimes AA_g,$$

then $\mathcal{A}$ is isomorphic to $1\otimes\mathcal{A}$. Hence we shall identify them. The *crossed product* $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G is the weak closure of $\mathcal{D}$ on $G\otimes\mathcal{H}$, that is, $G\otimes\mathcal{A}$ is the von Neumann algebra generated by $\mathcal{A}$ and $\{U_g; g\in G\}$ on $G\otimes\mathcal{H}$, where U_g is defined by

$$U_g(\Sigma_h\, h\otimes A_h)=\Sigma_h\, gh\otimes A_h^{g^{-1}},$$

for any $\Sigma_h\, h\otimes A_h\in\mathcal{D}$, being considered $\mathcal{D}$ as a dense linear subset of $G\otimes\mathcal{H}$.

3. At first we shall begin with the following

LEMMA 1. *Let $\mathcal{A}$ be the von Neumann algebra acting on a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ has the property Q and $\mathcal{A}'$ is finite, then there exists a linear functional τ defined on $\mathcal{L}(\mathcal{H})$, the algebra of all operators on $\mathcal{H}$, having the following properties:*

(a) $\tau(1)=1$,

(b) $\tau(A'T)=\tau(TA')$ *for* $A'\in\mathcal{A}'$ *and* $T\in\mathcal{L}(\mathcal{H})$,

(c) $\tau(T)\geqq 0$ *for* $T\geqq 0$ *and* $T\in\mathcal{L}(\mathcal{H})$,

(d) $\tau(T^*)=\tau(T)^*$ *for* $T\in\mathcal{L}(\mathcal{H})$,

(e) $\tau(U'TU'^*)=\tau(T)$ *for* $T\in\mathcal{L}(\mathcal{H})$,

where U' is a unitary operator of $\mathcal{A}'$.

Proof. Let $\mathcal{G}$ be the amenable generator of $\mathcal{A}$. For any operator $T\in\mathcal{L}(\mathcal{H})$, put

$$\sigma(T)=\int_{\mathcal{G}} UTU^*dU,$$

where the integration means the operator Banach mean on $\mathcal{G}$ in the sense of [1; § 2], then σ becomes a linear mapping from $\mathcal{L}(\mathcal{H})$ into $\mathcal{A}'$. By the assumption, $\mathcal{A}'$ is a finite von Neumann algebra, so that there exists a normalized trace ψ on $\mathcal{A}'$. Define $\tau(T)=\psi[\sigma(T)]$. We have shown in [1; Lemma 2] that the mapping τ satisfies the property (a), (b), (c), and (d). Since $\sigma(A'TB')=A'\sigma(T)B'$ for $A', B'\in\mathcal{A}'$, cf. [1; Lemma 1], $\tau(U'TU'^*)=\psi[\sigma(U'TU'^*)]=\psi[U'\sigma(T)U'^*]=\psi[\sigma(T)]=\tau(T)$, for any unitary operator $U'\in\mathcal{A}'$, whence (e) is proved.

THEOREM 1. *Let $\mathcal{A}$ be the von Neumann algebra of finite type, G an enumerable group of outer automorphisms on $\mathcal{A}$, and φ_0 the finite faithful normal trace on $\mathcal{A}$ which is invariant under G. If the crossed product $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G is the von Neumann algebra having the property Q, then G is an amenable group.*

To prove the theorem, we need the following

LEMMA 2. *Let $\mathcal{H}$ be the representation space of $\mathcal{A}$ by φ_0. Then there exists an operator $T_x\in\mathcal{L}(G\otimes\mathcal{H})$ for $x\in L^\infty(G)$ having the follow-*

ing properties:

(1) $T_{\alpha x+\beta y}=\alpha T_x+\beta T_y$, *where* α *and* β *are scalars*,

(2) $T_1=1$,

(3) $T_x \geqq 0$ *for* $x \geqq 0$,

(4) $U_g{}^* T_x U_g = T_{x_g}$ *for any* $g \in G$, *where* $x_g(h)=x(gh)$.

Proof. We define the operator T'_x on $\mathcal{D}$ (in §2) as follows: $T'_x(g\otimes A_g)=x(g)g\otimes A_g$ and $T'_x(\Sigma_g\, g\otimes A_g)=\Sigma_g\, T'_x(g\otimes A_g)$, for any $x\in L^\infty(G)$ and any $\Sigma_g g\otimes A_g \in \mathcal{D}$. This T'_x is clearly bounded, and can be extended to the operator T_x on $G\otimes\mathcal{H}$. It is plain by the definition that T_x satisfies the properties (2) and (3).

$$\begin{aligned} T_{\alpha x+\beta y}(\Sigma_g\, g\otimes A_g) &= \Sigma_g\,(\alpha x(g)+\beta y(g))g\otimes A_g \\ &= \alpha\Sigma_g\, x(g)g\otimes A_g+\beta\Sigma_g\, y(g)g\otimes A_g \\ &= (\alpha T_x+\beta T_y)\Sigma_g\, g\otimes A_g, \end{aligned}$$

for any $\Sigma_g\, g\otimes A_g \in \mathcal{D}$, whence (1) is proved. And

$$\begin{aligned} U_g^* T_x U_g(\Sigma_h\, h\otimes A_h) &= U_g^* T_x(\Sigma_h\, gh\otimes A_h^{g^{-1}}) \\ &= U_g^*(\Sigma_h\, x(gh)gh\otimes A_h^{g^{-1}}) \\ &= \Sigma_h\, x(gh)h\otimes A_h \\ &= T_{x_g}(\Sigma_h\, h\otimes A_h), \end{aligned}$$

for any $\Sigma_h\, h\otimes A_h \in \mathcal{D}$ and for any $g\in G$, whence (4) is proved.

Proof of Theorem 1. Since $G\otimes\mathcal{A}$ has the property Q, and $(G\otimes\mathcal{A})'$ is involutionally isomorphic to $G\otimes\mathcal{A}$ on $G\otimes\mathcal{H}$ by [8], $(G\otimes\mathcal{A})'$ has the property Q. Hence there exists a linear functional τ on $\mathcal{L}(G\otimes\mathcal{H})$ by Lemma 1. Put

$$\int_G x(g)dg=\tau(T_x) \quad \text{for any } x\in L^\infty(G).$$

Then it is sufficient to prove the theorem that $\int_G x(g)dg$ satisfies the following conditions:

$1°\quad \int[\alpha x(g)+\beta y(g)]dg=\alpha\int x(g)dg+\beta\int y(g)dg,$

$2°\quad \int x(g)dg\geqq 0$ if $x(g)\geqq 0$ for all $g\in G$,

$3'\quad \int x(hg)dg=\int x(g)dg,$

$4°\quad \int 1\, dg=1.$

$\int_G x(g)dg$ satisfies 1° by (1) and the linearity of τ. By (3) and (c), we have $\int_G x(g)dg\geqq 0$ for $x\geqq 0$, which shows 2°. By (4),

$$\int_G x(hg)dg=\tau(T_{x_h})=\tau(U_h^* T_x U_h)=\tau(T_x)=\int_G x(g)dg,$$

which proves 3'. Finally by (2),

$$\int_G 1\, dg = \tau(T_1) = \tau(1) = 1,$$

whence 4° is proved. Therefore G is amenable.

Since we have proved in [1; Theorem 5] that the hyperfinite factor has the property Q, Theorem 1 implies the following

COROLLARY. *If the crossed product $G \otimes \mathcal{A}$ of a von Neumann algebra of finite type by an enumerable group G of outer automorphisms of $\mathcal{A}$ is a continuous hyperfinite factor, then G is an amenable group.*

4. As a consequence of Theorem 1, we shall show at first the following theorem which may be considered as a reproduction of [4; Theorem 4]:

THEOREM 2. *If $\mathcal{A}$ is the continuous hyperfinite factor, then there exists a group G of outer automorphisms of $\mathcal{A}$ such that the crossed product $G \otimes \mathcal{A}$ is not hyperfinite.*

Proof. Since the free group Φ having two generators is isomorphically representable by the group of outer automorphisms of the hyperfinite factor $\mathcal{A}$ by [4; II, Theorem] and since Φ is not amenable, the theorem follows at once by Corollary.

Since Theorem 1 is still true if we replace the property Q by the property P of Schwartz [6], we can reproduce a proof by the above argument that there exists a continuous finite factor without the property P.

In the next place, we shall consider the example of factors due to Murray and von Neumann [5]:

THEOREM 3. *Let G be an enumerable ergodic m-group in a measure space (S, μ). If the factor constructed by G and (S, μ) by the method due to Murray and von Neumann is a continuous hyperfinite factor, then G is amenable.*

Proof. Let $\mathcal{A}$ be the multiplication algebra of the measure space (S, μ). Then G induces a group G_1 of outer automorphisms of $\mathcal{A}$ which is isomorphic to G and preserves the integral by μ. Turumaru [8] observed that the crossed product $G_1 \otimes \mathcal{A}$ is nothing but the factor constructed by Murray and von Neumann. Hence the theorem follows from Corollary.

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