103. On Wiener Homeomorphism between Riemann Surfaces

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1. Definition of Wiener homeomorphism (W.H.). In the theory of ideal boundaries of Riemann surfaces, the family of *Wiener functions* ([3], pp. 54–65) and that of *Dirichlet functions* ([3], pp. 65–85) are two main important classes of functions on Riemann surfaces. Let T be a homeomorphism of a Riemann surface R_1 onto another R_2. It is known ([4], [5]) that T is a general quasiconformal homeomorphism (which we shall abbreviate as Q.H.) of R_1 onto R_2 if and only if T preserves bounded continuous Dirichlet functions. In contrast with this, it is natural and has some interest to introduce a class of homeomorphisms between Riemann surfaces preserving bounded continuous Wiener functions. Let $\mathcal{W}(R)$ be the totality of *bounded continuous Wiener functions* on a Riemann surface R.

Definition. *A homeomorphism T of a Riemann surface R_1 onto another R_2 is called a Wiener homeomorphism (which we abbreviate as W.H.) of R_1 onto R_2 if $f\circ T$ belongs to $\mathcal{W}(R_1)$ when and only when f belongs to $\mathcal{W}(R_2)$.*

2. Algebraic and topological criterion of existence of W.H. Let R^* be the *Wiener compactification* ([3], pp. 96–109) of a Riemann surface R and $C(R^*)$ be the totality of real-valued bounded continuous functions on R^*. By definition, any function in $\mathcal{W}(R)$ can be continuously extended to R^* uniquely and so we may consider that $\mathcal{W}(R)\subset C(R^*)$. Since $\mathcal{W}(R)$ is a vector subspace of $C(R^*)$ which is closed under max and min operations ([3], p. 56) and $\mathcal{W}(R)$ separates points in R^* ([3], p. 98), by Stone's theorem ([3], p. 5), $\mathcal{W}(R)$ is dense in $C(R^*)$ with respect to the uniform convergence topology. Hence $\mathcal{W}(R)=C(R^*)$, since $\mathcal{W}(R)$ is uniformly closed. We call $\mathcal{W}(R)$ *Wiener algebra* on R in contrast with Royden algebra ([5]).

Theorem 1. *Any W.H. T of R_1 onto R_2 induces (and is induced by) an algebraic isomorphism $f\to f^\sigma$ of $\mathcal{W}(R_1)$ onto $\mathcal{W}(R_2)$ satisfying $f^\sigma=f\circ T^{-1}$.*[1]

Proof. We have only to show that any algebraic isomorphism $f\to f^\sigma$ of $\mathcal{W}(R_1)$ onto $\mathcal{W}(R_2)$ is induced by a W.H. T of R_1 onto R_2 with $f^\sigma=f\circ T^{-1}$. Since $\mathcal{W}(R_i)=C(R_i^*)$ and R_i^* is compact, any algebraic homomorphism of $\mathcal{W}(R_i)$ onto real numbers is of the form $f\to f(p)$, where p is a unique fixed point in R_i^* determined by this homomorphism. Let $p\in R_1^*$. Then $f\to f^{\sigma^{-1}}(p)$ is an algebraic homomorphism of $\mathcal{W}(R_2)$ onto real numbers and so there exists a unique point $T^*(p)$ in R_2^* such that $f^{\sigma^{-1}}(p)=f(T^*(p))$. From this, it is easy to see that T^* is a homeomorphism of R_1^* onto R_2^*. Let $p\in R_1$. Then since any point in $R_2^*-R_2$ cannot have a countable fundamental neighborhood system ([3], p. 103), $T^*(p)$ must belong to R_2. Thus the restriction of T^* on R_1 gives rise to a homeomorphism T of R_1 onto R_2 and $f^{\sigma^{-1}}=f\circ T$ on R_1 or $f^{\sigma}=f\circ T^{-1}$ on R_2 assures that T is a W.H. of R_1 onto R_2.

Theorem 2. *Any W.H. T of R_1 onto R_2 can be extended to a homeomorphism T^* of R_1^* onto R_2^**[1] *and conversely, the restriction on R_1 of any homeomorphism T^* of R_1^* onto R_2^* gives rise to a W.H. T of R_1 onto R_2.*[2]

Proof. Let T be a W.H. of R_1 onto R_2. Then $f\to f\circ T^{-1}$ induces an algebraic isomorphism of $\mathcal{W}(R_1)$ onto $\mathcal{W}(R_2)$ and as in the proof of Theorem 1, this isomorphism induces a homeomorphism T^* of R_1^* onto R_2^* such that $f\circ T^{-1}=f\circ T^{*^{-1}}$ for any f in $\mathcal{W}(R_2)=C(R_2^*)$. Thus the restriction of T^* on R_1 is T. Conversely, assume that T^* is a homeomorphism of R_1^* onto R_2^*. Then $f\to f\circ T^{*^{-1}}$ induces an algebraic isomorphism of $C(R_1^*)=\mathcal{W}(R_1)$ onto $C(R_2^*)=\mathcal{W}(R_2)$ and so by Theorem 1, there exists a W.H. T of R_1 onto R_2 such that $f\circ T^{*^{-1}}=f\circ T^{-1}$ for any f in $C(R_2^*)=\mathcal{W}(R_2)$. Thus the restriction of T^* on R_1 is a W.H. T of R_1 onto R_2.

3. Absolute continuity of W.H. on Wiener boundary. We denote by Δ the *Wiener boundary* R^*-R and by Γ (Wiener) *harmonic boundary* of R^* ([3], p. 90). The set Δ (resp. Γ) is a compact subset of R^* (resp. Δ). If we denote by $\mathcal{W}_0(R)$ the totality of bounded continuous Wiener potentials ([3], p. 56), then $\Gamma=(p\in R^*;\ f(p)=0$ for any f in $\mathcal{W}_0(R))$ and $\mathcal{W}_0(R)=(f\in\mathcal{W}(R);\ f=0$ on $\Gamma)$.

Theorem 3. *Any W.H. T of a Riemann surface R_1 onto R_2 can be extended to a homeomorphism T^* of R_1^* onto R_2^* and $T^*(\Gamma_1)=\Gamma_2$.*[1]

Proof. Let $p_1\in\Gamma_1$. Clearly $p_2=T^*(p_1)\in\Delta_2$. We have to show that $p_2\in\Gamma_2$. Contrary to the assertion, assume that $p_2\in\Delta_2-\Gamma_2$. Since Γ_2 is compact, we can find two open neighborhoods F_2^* and G_2^* of p_2 such that $F_2^*\supset\overline{G_2^*}$ and $\overline{F_2^*}\frown\Gamma_2=\phi$. Moreover we may assume that relative boundaries of $F_2=F_2^*\frown R_2$ and $G_2=G_2^*\frown R_2$ consist of at most countably many piecewise analytic Jordan curves not ending and not accumulating in R_2. We set $F_1^*=T^{*^{-1}}(F_2^*)$, $G_1^*=T^{*^{-1}}(G_2^*)$, $F_1=T^{-1}(F_2)=F_1^*\frown R_1$ and $G_1=T^{-1}(G_2)=G_1^*\frown R_1$. Since $\Gamma_1\frown\overline{R_1-G_1}\not\ni p_1$, $\Gamma_1\not\subset\overline{R_1-G_1}$ and so there exists a connected component of G_1, say G_1', not of type SO_{HB} ([3], Satz 9.12, p. 108).

1) The same is true for Q.H. and Royden's compactification and algebra ([5], [4]).

2) This is not true for Q.H. and Royden's compactification.

Let $G_2'=T(G_1')$. We can find a normal exhaustion $(R_n^{(2)})_{n=1}^{\infty}$ of R_2 such that $\partial R_n^{(2)}\frown G_2'\neq\phi$. Then $(R_n^{(1)})_{n=1}^{\infty}$ is an exhaustion of R, where $R_n^{(1)}=T^{-1}(R_n^{(2)})$. We can find a real-valued continuous function f_2 on R_2 such that $0\leq f_2\leq 1$ on R_2 and $f_2=0$ on $\bigcup_{n=1}^{\infty}(\partial R_{2n}^{(2)}\frown\overline{F_2^*}\frown R_2)\smile(R_2-F_2)$ and $f_2=1$ on $\bigcup_{n=1}^{\infty}(\partial R_{2n-1}^{(2)}\frown\overline{G_2^*}\frown R_2)$. Let $f_1=f_2\circ T$. Then f_1 is continuous function on R_1 such that $0\leq f_1\leq 1$ on R_1 and $f_1=0$ on $\bigcup_{n=1}^{\infty}\partial R_{2n}^{(1)}$ and $f_1=1$ on $\bigcup_{n=1}^{\infty}(\partial R_{2n-1}^{(1)}\frown G_1')$. As $G_1'\notin SO_{HB}$, so there exists a continuous function h on R_1 such that $0\leq h\leq 1$ on R_1 and $h=0$ on R_1-G_1' and $h\in HB(G_1')$ and $h>0$ on G_1'. Clearly $H_{f_1}^{R_{2n}^{(1)}}=0$ and $H_{f_1}^{R_{2n-1}^{(1)}}>h>0$. Thus $\lim_{n\to\infty}H_{f_1}^{R_n^{(1)}}$ does not exist on R_1. Hence $f_1\notin\mathcal{W}(R_1)$ ([3], Satz 6.2, p. 57). Let $K=\overline{F_2^*}\frown\Delta_2$. Then there exists a positive finite superharmonic function S on R_2 such that $\lim_{R\ni z\to p}S(z)=\infty$ for any p in K. Let $(R_n')_{n=1}^{\infty}$ be an arbitrary exhaustion of R_2 and $\varepsilon>0$. Then $0\leq H_{f_2}^{R'_n}<\varepsilon S$ for sufficiently large n. Thus $\limsup_{n\to\infty}H_{f_2}^{R_n^{(2)}}(z)\leq\varepsilon S(z)$ for any point z in R_2, or $\lim_{n\to\infty}H_{f_2}^{R_n'}\equiv 0$ on R_2. This shows that $f_2\in\mathcal{W}(R_2)$ ([3], Satz 6.3, p. 62). Thus we have $f_1=f_2\circ T$ and $f_1\notin\mathcal{W}(R_1)$ and $f_2\in\mathcal{W}(R_2)$. This contradicts the fact that T is a W.H.

Corollary 3.1. *Let T be a W.H. of a hyperbolic Riemann surface R_1 onto another R_2 and G be a subdomain of R_1. Then $G\in SO_{HB}$ if and only if $TG\in SO_{HB}$.*

Proof. $G\in SO_{HB}$ if and only if $\Gamma_1\subset\overline{R_1-G}$ ([3], p. 108). This with Theorem 3 proves our assertion.

Theorem 4. *Let T be a W.H. of a Reimann surface R_1 onto R_2 and T^* be its homeomorphic extension of R_1^* onto R_2^*. The set X in Δ_1 is of harmonic measure zero ([3], p. 87) if and only if $T^*(X)$ is of harmonic measure zero in Δ_2.*[3]

Proof. Let ω_i be the harmonic measure on Δ_i. Since $\omega_i(\Delta_i-\Gamma_i)=0$, we have only to prove that if the set X in Γ_1 is of harmonic measure zero, then $T^*(X)$ is of harmonic measure zero. As Γ_1 is a Stonean space ([3], p. 101), we can find a sequence $(K_n)_{n=1}^{\infty}$ of open and compact subsets K_n in Γ_1 such that $K_1\supset K_2\supset\cdots\supset K_n\supset X$ and $\omega_1(K_n)\searrow 0$ $(n\to\infty)$. By Theorem 3, $T^*(X)\subset\Gamma_2$. Contrary to the assertion, assume that $\omega_2(T^*X)>0$. Let f_n be the characteristic function of T^*K_n. Then f_n is continuous on Γ_2 and $H_{f_n}^{R_2}(z)\geq H_{f_{n+p}}^{R_2}(z)=\int f_{n+p}d\omega_{2,z}=\omega_{2,z}(T^*K_{n+p})\geq\omega_{2,z}(T^*X)\gneqq 0$. Hence $u(z)=\lim_n H_{f_n}^{R_2}(z)$ is a strictly positive HB-function on R_2 with $u\leq H_{f_n}^{R_2}$ on R_2. Thus $U=(p\in\Gamma_2;u(p)>0)$ is a non-void open set in Γ_2. As $f_n(q)=H_{f_n}^{R_2}(q)\geq u(q)$ on Γ_2 ([3], p. 101), so $U\subset T^*K_n$. By the fact that Γ_2 is a Stonean space, $\overline{U}(\subset T^*K_n)$ is

3) This is not true for Q.H. and Royden's compactification (see [6], p. 175).

open and closed and so is $K=T^{*^{-1}}(\overline{U})$ in Γ_1 and $K_n \supset K$ $(n=1,2,\cdots)$. Let f be the characteristic function of K on Γ_1. Then f is continuous on Γ_1 and so $H_f^{R_1}$ is a non-negative HB-function on R_1 and so continuous on R_1^* and $H_f^{R_1}(p)=f(p)$ on Γ_1. Since $f(p)>0$ and $f(p)\not\equiv 0$ on Γ_1, $H_f^{R_1}$ is strictly positive on R_1. Thus for any point z in R_1, $0<H_f^{R}(z)=\int f d\omega_{1,z}=\omega_{1,z}(K)$, i.e. $\omega_1(K)>0$. Hence $\omega_1(K_n)\geq\omega_1(K)>0$ $(n=1,2,\cdots)$. This contradicts the fact that $\omega_1(K_n)\searrow 0$ $(n\to\infty)$.

4. Invariance of some classes of open Riemann surfaces by W. H.

Theorem 5. *O_G, O_{HB}, O_{HB}^n $(1\leq n\leq\infty)$ and U_{HB} are invariant by W. H.*[4]

Proof. Let R be an open Riemann surface. $R\in O_G$ if and only if $\Gamma=\phi$, $R\in O_{HB}$ if and only if Γ consists of only one point, $R\in O_{HB}^n - O_{HB}^{n-1}$ $(2\leq n<\infty)$ if and only if Γ consists of n points, $R\in O_{HB}-\bigcup_{n=1}^{\infty}O_{HB}^n$ if and only if $\Gamma=(p_1,p_2,\cdots)\cup X$ with $\omega(p_i)>0$ $(n=1,2,\cdots)$ and $\omega(X)=0$, $R\in U_{HB}$ if and only if Γ contains a point p with $\omega(p)>0$ ([3], pp. 125–127). From this with Theorems 3 and 4, we get our assertion.

5. W. H. of open unit disc. Let $U=(z;\ |z|<1)$, $C=\partial U$ and $\widetilde{U}=U\cup C$.

Theorem 6. *Let T be a W. H. of U onto U. Then T can be continuously extended so as to be a homeomorphism $\widetilde{T}$ of $\widetilde{U}$ onto $\widetilde{U}$.*[5]

Proof. By applying a suitable linear transformation, we may assume that $T(0)=0$. Let $\zeta\in C$ and $C_U(T,\zeta)$ be the cluster set of T at ζ in U. First we show that $C_U(T,\zeta)$ consists of only one point in C. If this is not the case, then $C_U(T,\zeta)$ is a non-degenerated closed subarc A of C. Let $V_n=(z\in\widetilde{U};\ |z-\zeta|\leq 1/n)$ $(n=1,2,\cdots)$ and f_n be a continuous function on U such that $f_n=1$ on V_n and f_n is harmonic in $U-V_n$ with boundary values 0 at $C-V_n$ and 1 at ∂V_n. Then f_n is a superharmonic in U and so $f_n\in\mathcal{W}(U)$ and also $g_n=f_n\circ T^{-1}\in\mathcal{W}(U)$. We decompose f_n and g_n into the forms $f_n=u_n+\varphi_n$ and $g_n=v_n+\phi_n$, where u_n, $v_n\in HB(U)$ and φ_n, $\phi_n\in\mathcal{W}_0(U)$. Let T^* be the continuous extension of T of the Wiener compactification U^* of U onto U^*. Then $g_n\circ T^*=g_n\circ T=f_n$ on U and so $g_n\circ T^*=f_n$ on U^*. Since T^* preserves the harmonic boundary Γ of U and $\varphi_n=\phi_n=0$ on Γ, we get that $u_n=v_n\circ T^*$ on Γ. Clearly, u_n is the harmonic measure of $V_n\cap C$ in U and so $u_n\wedge(1-u_n)=0$ in U. Hence $\min(u_n(p),\ 1-u_n(p))=0$ on Γ ([3], p. 56). Thus u_n takes only two values 0 and 1 on Γ and the same is true for v_n, since $u_n=v_n\circ T^*$ on Γ and $T^*(\Gamma)=\Gamma$. Let $K_n=(p\in\Gamma;\ u_n(p)=1)$. Then $T^*(K_n)=(p\in\Gamma;\ v_n(p)=1)$. Let ω be the

4) Compare this with the result of Pfluger [7] and Royden [8] concerning Q. H.

5) This is true for Q. H. (see [1]).

harmonic measure on Γ with respect to 0. Then $\omega(K_n)=\int_\Gamma u_n(p)d\omega(p)=u_n(0)$ and similarly, $\omega(T^*(K_n))=v_n(0)$. Clearly, $u_n(0)\searrow 0$ on U and so $K_1\supset K_2\supset K_3\cdots$ and so $T^*(K_1)\supset T^*(K_2)\supset T^*(K_3)\supset\cdots$ and $\omega(K_n)\searrow 0$. Thus $\omega(\bigcup_{n=1}^\infty K_n)=\lim_n \omega(K_n)=0$ and by Theorem 4, $0=\omega(T^*(\bigcup_{n=1}^\infty K_n))=\omega(\bigcup_{n=1}^\infty T^*(K_n))=\lim_n \omega(T^*(K_n))=\lim_n v_n(0)$. Thus $v_n(0)\searrow 0$. But this is a contradiction. In fact, there exists a bounded Green potential S in U such that $|\phi_n|\leq S$ in U ([3], Hilfssatz 6.4, p. 56). Then by Littlewood's theorem (see for example, [9], Theorem IV. 33 in p. 170 and Theorem IV. 34 in p. 172), $\lim_{r\nearrow 1} S(re^{i\theta})=0$ and so $\lim_{r\nearrow 1}\phi_n(re^{i\theta})=0$ for almost every $e^{i\theta}$ in C. As $v_n(e^{i\theta})=\lim_{r\nearrow 1}v_n(re^{i\theta})$ exists for almost every $e^{i\theta}$ in C and so $\lim_{r\nearrow 1} g_n(re^{i\theta})$ exists and equals $v_n(e^{i\theta})$ for almost every $e^{i\theta}$ in C. Let $\gamma_w=(re^{i\arg w};\ 0\leq r<1)$ and $\gamma'_w=T^{-1}(\gamma_w)$. We see that the closure $\tilde{\gamma}'_w$ of γ'_w in $\tilde{U}$ contains ζ for each w in the interior of the arc $A=C_U(T,\zeta)$ except at most one w in it. To see this, assume that there exist two distinct points w_1 and w_2 in the interior of the arc A such that $\zeta\notin\tilde{\gamma}'_{w_1}\cup\tilde{\gamma}'_{w_2}$. As $\gamma_{w_1}\cup\gamma_{w_2}$ divides U into two components U_1 and U_2, so $\gamma'_{w_1}\cup\gamma'_{w_2}$ divides U into two components U'_1 and U'_2. We assume that $T(U'_i)=U_i$ $(i=1,2)$. Since $\gamma'_{w_1}\cup\gamma'_{w_2}=\partial U'_i$ $(i=1,2)$ is free from ζ, one of the closures $\tilde{U}'_i$ of U'_i in U $(i=1,2)$, say $\tilde{U}'_1$, is a neighborhood of ζ in $\tilde{U}$. Then $A=C_U(T,\zeta)\subset\widetilde{T(U'_1)}=U_1$ by the definition of the cluster set and so $A\frown(C-\tilde{U}_1)=\phi$, which is clearly a contradiction, since w_1 and w_2 are contained in the interior of A and so $A\frown(C-\tilde{U}_1)\neq\phi$. Thus $\zeta\in\tilde{\gamma}'_w$ for each w in A except at most three points in A. Let A_n be the set of w in A such that g_n has the limit along γ_w and $\zeta\in\tilde{\gamma}'_w$. Then $A-A_n$ is of linear measure zero. For each $w\in A_n$, we can find a sequence $z_m\in\gamma'_w$ such that $z_m\to\zeta$ in $\tilde{U}$. Then $T(z_m)=r_m e^{i\arg w}\to w$. Thus $v_n(e^{i\arg w})=\lim_{r\nearrow 1} g_n(re^{i\arg w})=\lim_m g_n(r_m e^{i\arg w})=\lim_m g_n(T(z_m))=\lim_m f_n(z_m)=1$. Hence $v_n(e^{i\theta})=1$ for all $e^{i\theta}$ in A_n and so for almost every $e^{i\theta}$ in A. Since $v_n(z)\geq 0$ on U, $v_n(0)=(1/2\pi)\int_0^{2\pi} v_n(e^{i\theta})d\theta\geq(1/2\pi)\int_{A_n} d\theta=(1/2\pi)\int_A d\theta$. Hence $0=\lim_n v_n(0)\geq(1/2\pi)\int_A d\theta>0$, a contradiction.

Thus $C_U(T,\zeta)$ consists of one point in C, say $\tilde{T}\zeta$, for any ζ in C. Thus by setting $\tilde{T}(z)=T(z)$ for z in U, $\tilde{T}$ is a continuous mapping of $\tilde{U}$ onto $\tilde{U}$. Assume that there exist two distinct points b_1 and b_2 in C such that $\tilde{T}(b_1)=\tilde{T}(b_2)$. We take an analytic Jordan arc L in U connecting b_1 and b_2. Then $\overline{T(L)}$ is a closed Jordan curve in U with $\overline{T(U)}\frown C=T(b_1)=T(b_2)$. The interior G_2 of $\overline{T(L)}$ is a subdomain of U and $G_1=T^{-1}(G_2)$ is also a subdomain of U with $\partial G_1=L$. Clearly $G_1\notin SO_{HB}$

and $G_2 \in SO_{HB}$. This contradicts Corollary 3.1.

Theorem 7. *Let T be a W.H. of U onto U and $\widetilde{T}$ be its homeomorphic extension of $\widetilde{U}$ onto $\widetilde{U}$. Then $\widetilde{T}$ is an absolutely continuous homeomorphism of C onto C.*[6]

Proof. The identity map of U onto U can be extended to a continuous mapping ρ of U^* onto $\widetilde{U}$ uniquely ([3], p. 99). Notice that $(2\pi)^{-1}d\theta$ is the harmonic measure on C with the reference point 0. Hence for any bounded continuous function f on C, we get $\int_{\Delta} f\circ\rho\, d\omega^{7)}=(2\pi)^{-1}\int_C f\, d\theta$ ([3], Satz 8.6, p. 92). From this, for any compact set K, it follows easily that $\omega(K)=(2\pi)^{-1}\int_K d\theta$.

It is easy to see that $\rho\circ T^*=\widetilde{T}\circ\rho$ in U^* and similarly $\rho\circ(T^{-1})^*=(\widetilde{T^{-1}})\circ\rho$ in U^*. Let F be a compact set in C with $\int_F d\theta=0$. We have to show that $\int_{\widetilde{T}(F)} d\theta=0$. By the above, $\omega(\rho^{-1}(F))=(2\pi)^{-1}\int_F d\theta=0$ and so $\omega(T^*(\rho^{-1}(F)))=0$ by Theorem 4. On the other hand, $\widetilde{T}(F)=\widetilde{T}(\rho(\rho^{-1}(F)))=(\widetilde{T}\circ\rho)(\rho^{-1}(F))=(\rho\circ T^*)(\rho^{-1}(F))$, i.e. $\rho\circ(T^*(\rho^{-1}(F)))=\widetilde{T}(F)$. Hence $T^*(\rho^{-1}(F))\subset\rho^{-1}(\widetilde{T}(F))$. Let $q\in\rho^{-1}(\widetilde{T}(F))$ and $p=(T^{-1})^*(q)=(T^*)^{-1}(q)$. Then $\rho(p)=(\rho\circ(T^{-1})^*)(q)=((\widetilde{T^{-1}})\circ\rho)(q)=(\widetilde{T})^{-1}(\rho(q))$. As $\rho(q)\in\widetilde{T}(F)$, so $(\widetilde{T})^{-1}(\rho(q))\in(\widetilde{T})^{-1}(\widetilde{T}(F))=F$. Thus $p\in\rho^{-1}(F)$ and so $q=T^*((T^*)^{-1}(q))=T^*(p)\in T^*(\rho^{-1}(F))$. Thus we conclude that $T^*(\rho^{-1}(F))=\rho^{-1}(\widetilde{T}(F))$. Hence $\int_{\widetilde{T}(F)} d\theta=(2\pi)\omega(\rho^{-1}(\widetilde{T}(F))=(2\pi)\omega(T^*(\rho^{-1}(F))=0$.

References

[1] L. V. Ahlfors: On quasiconformal mappings. J. d'Analyse Math., **3**, 1–58, 207–208 (1953/4).

[2] A. Beurling and L. V. Ahlfors: The boundary correspondence under quasiconformal mappings. Acta Math., **96**, 125–142 (1956).

[3] C. Constantinescu and A. Cornea: Ideale Ränder Riemannscher Flächen. Springer-Verlag, Berlin-Göttingen-Heidelberg (1963).

[4] M. Nakai: On a ring isomorphism induced by quasiconformal mappings. Nagoya Math., J., **14**, 201–221 (1959).

[5] ——: Algebraic criterion on quasiconformal equivalence of Riemann surfaces. Nagoya Math. J., **16**, 157–184 (1960).

[6] ——: Genus and classification of Riemann surfaces. Osaka Math. J., **14**, 153–180 (1962).

[7] A. Pfluger: Sur un propriété de l'application quasi conforme d'une surface de Riemann ouverte. C. R. Acad. Sci. Paris, **227**, 25–26 (1948).

[8] H. L. Royden: A property of quasi-conformal mapping. Proc. Amer. Math. Soc., **5**, 266–269 (1954).

[9] M. Tsuji: Potential Theory in Modern Function Theory. Maruzen, Tokyo (1959).

6) This is not true for Q.H. by virtue of the important example of Beurling-Ahlfors [2].

7) We assume that the reference point of ω is 0.