# 140. *Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XIV*

By Sakuji INOUE

Faculty of Education, Kumamoto University

(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1964)

Let $T(\lambda)$ be such a function as was defined in the preceding paper (that is, in Part XIII) [cf. Proc. Japan Acad., Vol. 40, No. 7 (1964)]. In the present paper we shall derive another formula of the expansion of $T(\lambda)$ and shall discuss some of its applications.

Since, as we have already shown in Part XIII, the form of the expansion by series or by integrals of $T(\lambda)$ outside a suitably large circle with center at the origin is exactly similar to that of the function $S(\lambda)$ in Theorem 1 [cf. Proc. Japan Acad., Vol. 38, No. 6, 265–267 (1962)], we can establish the following propositions for the question as to whether the ordinary part of $T(\lambda)$ is a polynomial in $\lambda$ or a transcendental integral function.

Proposition A. Let $T(\lambda)$, $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$, $D_j$, $(j=1,2,3,\cdots,n)$, and $\rho$ be the same notations as those used in Theorem 33 respectively, and $M_T(\rho,0)$ the maximum modulus of $T(\lambda)$ for all the points on the circle $|\lambda|=\rho$. Then a necessary and sufficient condition that the ordinary part of $T(\lambda)$ be a polynomial in $\lambda$ of the degree less than or equal to $d$ is that there exist a positive constant $K$ and a suitably large number $\sigma$ such that

$$M_T(r,0)\leqq Kr^d$$

for every $r$ with $\max\,[\sup_\nu|\lambda_\nu|,\ \max_j(\max_{z\in D_j}|z|)]<\sigma<r<\infty$ [cf. Proc. Japan Acad., Vol. 38, No. 10, 706–707 (1962)].

Proposition B. Let $T(\lambda)$, $\rho$, and $M_T(\rho,0)$ be the same notations as above respectively. Then a necessary and sufficient condition that the ordinary part of $T(\lambda)$ be a transcendental integral function of the order $d>0$ is that

$$\overline{\lim_{\rho\to\infty}}\frac{\log\log M_T(\rho,0)}{\log\rho}=d>0 \quad \text{[cf. loc. cit., 708–709].}$$

In fact, these propositions can be shown by replacing $S(\lambda)$ in the proofs of Theorems 13 and 14 by $T(\lambda)$. As for Proposition A, however, we can simplify it by making use of the following theorem derived from the already established expansion by series of $T(\lambda)$ outside a suitably large circle with center at the origin.

Theorem 38. Let $T(\lambda)$ and $\rho$ be the same notations as those used in Theorem 33 respectively, and let

$$c_k=\frac{1}{2\pi i}\int_{|\lambda|=\rho}\frac{T(\lambda)}{\lambda^{k+1}}d\lambda \quad (k=0, \pm 1, \pm 2, \cdots, i=\sqrt{-1}),$$

where the circle $|\lambda|=\rho$ is positively oriented. Then

$$T(\lambda)=\sum_{p\geqq 0} c_p\lambda^p+\sum_{p=1}^{\infty} c_{-p}\lambda^{-p} \quad (\rho\leqq|\lambda|<\infty); \tag{33}$$

and $\sum_{p\geqq 0} c_p\lambda^p$ with domain $\{\lambda: |\lambda|<\infty\}$ extended to the origin expresses the ordinary part $R(\lambda)$ of $T(\lambda)$ and is a finite or an infinite series according as $R(\lambda)$ is a polynomial in $\lambda$ or a transcendental integral function, while $\sum_{p=1}^{\infty} c_{-p}\lambda^{-p}$ expresses the sum of the first and the second principal parts of $T(\lambda)$ in the domain $\{\lambda: \rho\leqq|\lambda|\}$ extended to the point $\lambda=\infty$ and is essentially an infinite series.

Proof. Let $\chi(\lambda)$ denote the sum of the first and the second principal parts of $T(\lambda)$, and let

$$\begin{cases} a_p=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} T(\rho e^{it}) \cos pt\, dt & (p=0, 1, 2, \cdots) \\ b_p=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} T(\rho e^{it}) \sin pt\, dt & (p=1, 2, 3, \cdots). \end{cases}$$

Then, as already shown in the preceding paper,

$$T\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{a_0}{2}+\frac{1}{2}\sum_{p=1}^{\infty}(a_p-ib_p)\left(\frac{e^{i\theta}}{\kappa}\right)^p+\frac{1}{2}\sum_{p=1}^{\infty}(a_p+ib_p)\left(\frac{\kappa}{e^{i\theta}}\right)^p \quad (0<\kappa<1),$$

where

$$\frac{a_0}{2}+\frac{1}{2}\sum_{p=1}^{\infty}(a_p-ib_p)\left(\frac{e^{i\theta}}{\kappa}\right)^p=R\left(\frac{\rho e^{i\theta}}{\kappa}\right)$$

for every $\kappa$ with $0<\kappa<\infty$ and for $\kappa$ becoming infinite and

$$\frac{1}{2}\sum_{p=1}^{\infty}(a_p+ib_p)\left(\frac{\kappa}{e^{i\theta}}\right)^p=\chi\left(\frac{\rho e^{i\theta}}{\kappa}\right)$$

for every $\kappa$ with $0<\kappa<1$ and for $\kappa$ tending to zero. On the other hand,

$$a_p-ib_p=\frac{1}{\pi}\int_0^{2\pi} T(\rho e^{it})e^{-ipt}dt=\frac{\rho^p}{\pi i}\int_{|\lambda|=\rho}\frac{T(\lambda)}{\lambda^{p+1}}d\lambda$$

and similarly

$$a_p+ib_p=\frac{1}{\rho^p\pi i}\int_{|\lambda|=\rho}\frac{T(\lambda)}{\lambda^{-p+1}}d\lambda.$$

These results enable us to assert that

$$T(\lambda)=\sum_{p\geqq 0} c_p\lambda^p+\sum_{p=1}^{\infty} c_{-p}\lambda^{-p} \quad (\rho<|\lambda|<\infty).$$

In addition, since by Cauchy's integral theorem we have

$$\frac{1}{2\pi i}\int_{|\lambda|=\rho}\frac{T(\lambda)}{\lambda^{k+1}}d\lambda=\frac{1}{2\pi i}\int_{|\lambda|=\rho'}\frac{T(\lambda)}{\lambda^{k+1}}d\lambda \quad (k=0, \pm 1, \pm 2, \cdots)$$

where $\rho'$ is a positive constant less than $\rho$ such that the union of

$\bigcup_{j=1}^{n} D_j$ and the closure of $\{\lambda_\nu\}$ is wholly contained within the circle $|\lambda|=\rho'$ oriented positively, it turns out that the just established relation is valid in the domain $\{\lambda: \rho'<|\lambda|<\infty\}$. Thus it is found immediately from the above argument that the results in the statement of the present theorem are valid, except that the second member on the right of (33) is surely an infinite series.

If we now denote by $\rho_0$ the greatest lower bound of all positive numbers $\rho'$ satisfying the above-mentioned condition, it is seen that the relation (33) holds for any fixed point $\lambda$ with $\rho_0<|\lambda|<\infty$ and that there exists on the circle $|\lambda|=\rho_0$ at least one point $\xi$ belonging to the union of $\bigcup_{j=1}^{n} D_j$ and the closure of $\{\lambda_\nu\}$. Since, on the other hand, $T(\lambda)$ is expressible in the form of

$$T(\lambda)=R(\lambda)+\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}(f_{1\alpha}+f_{2\alpha}),(f'_{1\alpha}+f'_{2\alpha})) + \sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta}, g'_{j\beta}) \tag{34}$$

as stated in the preceding paper, and since, for any positive integer $p$,

$$(\lambda I-N_j)^{-p}=\int_{|z|\leq\|N_j\|}\frac{1}{(\lambda-z)^p}dK_j(z) \quad (j=1,2,3,\cdots,n)$$

where $\{K_j(z)\}$ denotes the complex spectral family of the bounded normal operator $N_j$, $T(\lambda)$ cannot be bounded in the intersection of the domain $\{\lambda: \rho_0<|\lambda|<\infty\}$ and a suitably small neighbourhood of $\xi$. Consequently it never occurs that $c_{-p}$ vanishes for every positive integer $p$ greater than some positive constant; that is, $\sum_{p=1}^{\infty}c_{-p}\lambda^{-p}$ is certainly an infinite series: for otherwise $T(\lambda)$ would be bounded in the intersection stated above and hence $\xi$ would be a point belonging to the resolvent set of every $N_j$ $(j=1,2,3,\cdots,n)$, contrary to the hypothesis on $\xi$.

The theorem has thus been proved.

Theorem 39. Let $T(\lambda)$ be the function in Theorem 38. Then a necessary and sufficient condition that the ordinary part of $T(\lambda)$ be a polynomial in $\lambda$ of the degree $d$ is that $T(\lambda)/\lambda^d$ tend to a non-zero finite constant as $|\lambda|\to\infty$.

Proof. It is at once obvious by Theorem 38 that the condition is necessary. If we denote the ordinary part and the sum of the two principal parts of $T(\lambda)$ by $R(\lambda)$ and $\chi(\lambda)$ respectively, as before, and assume that conversely $T(\lambda)/\lambda^d$ tends to a non-zero finite constant $C$ as $|\lambda|$ becomes infinite, then $\lim_{|\lambda|\to\infty} R(\lambda)/\lambda^d=C$ in accordance with Theorem 38. We now suppose that $\sum_{p\geqq d}c_p\lambda^{p-d}$ is an infinite power series, with a view of establishing a contradiction. Then $\sum_{p=d}^{\infty}c_p\lambda^{p-d}\to C$ $(|\lambda|$

$\to\infty$) and the function defined by this infinite power series $\sum_{p=d}^{\infty} c_p\lambda^{p-d}$ with domain $\{\lambda: |\lambda|<\infty\}$ would be a transcendental integral function with isolated essential singularity $\lambda=\infty$. Denoting by $\omega$ a complex finite constant different from $C$ and two exceptional values at most of this transcendental integral function, it is therefore found by Picard's theorem that there would exist an infinite sequence of complex numbers $z_1, z_2, z_3, \cdots$ outside a suitably large circle with center at the origin such that

$$\sum_{p=d}^{\infty} c_p z_\mu^{p-d}=\omega \quad (\mu=1, 2, 3, \cdots),$$

where $|z_1|<|z_2|<\cdots$ and $|z_\mu|\to\infty$ $(\mu\to\infty)$. This result is, however, in contradiction with the fact that $\sum_{p=d}^{\infty} c_p\lambda^{p-d}\to C$ $(|\lambda|\to\infty)$. In consequence, the supposition on $\sum_{p\geqq d} c_p\lambda^{p-d}$ must be rejected; that is, the condition is sufficient as we wished to prove.

Remark. The right-hand member of (33) converges absolutely and uniformly in the domain $(\lambda: \rho\leqq|\lambda|\leqq R<\infty\}$, however large $R$ may be.

Theorem 40. Let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be an arbitrarily prescribed bounded infinite sequence of complex numbers such that its closure never contains zero; let $\varDelta_1, \varDelta_2, \cdots, \varDelta_{n-1}$, and $\varDelta_n$ be arbitrarily given, connected, and closed sets in the complex plane such that one of them is not necessarily bounded, that each of them does not contain the origin and any point belonging to the closure of $\{\lambda_\nu\}$, and that they are mutually disjoint; let $\rho$ be an arbitrarily given positive constant such that the union of $\bigcup_{j=1}^{n}\varDelta_j$ and the closure of $\{\lambda_\nu\}$ lies in the domain $\{\lambda: \rho<|\lambda|\}$; and let $\tilde{T}(\lambda)$ be a function satisfying the following conditions:

(i) $\{\lambda_\nu\}$ is the set of all non-zero poles of $\tilde{T}(\lambda)$ in the sense of the functional analysis and the principal part of $\tilde{T}(\lambda)$ at $\lambda_\nu$ is given by $\sum_{\alpha=1}^{m} c_\alpha^{(\nu)}/(\lambda-\lambda_\nu)^\alpha$ where $\sum_{\nu=1}^{\infty}|c_\alpha^{(\nu)}|<\infty$ for $\alpha=1, 2, 3, \cdots, m$;

(ii) every point of $\bigcup_{j=1}^{n}\varDelta_j$ is a singularity of $\tilde{T}(\lambda)$ in the single meaning that the image of $\bigcup_{j=1}^{n}\varDelta_j$ by the transformation $1/\lambda$ determines the second principal part of $\tilde{T}(1/\lambda)$ alone;

(iii) the origin is a pole or an isolated essential singularity of $\tilde{T}(\lambda)$;

(iv) $\tilde{T}(\lambda)$ is regular in the entire complex $\lambda$-plane (inclusive of $\lambda=\infty$) with the exception of the union of the origin, $\bigcup_{j=1}^{n}\varDelta_j$, and the closure of $\{\lambda_\nu\}$;

(v) $\tilde{T}(\lambda)$ has no term with isolated essential singularity in the domain $\{\lambda: 0<|\lambda|\}$.

Then, in the domain $\{\lambda: 0<|\lambda|\leqq\rho\}$, $\tilde{T}(\lambda)$ is expressible in the form of

$$\tilde{T}(\lambda)=\sum_{p\geqq 0} c_p\lambda^{-p}+\sum_{p=1}^{\infty} c_{-p}\lambda^p, \tag{35}$$

where

$$c_k=\frac{1}{2\pi i}\int_{|\lambda|=1/\rho}\frac{\tilde{T}(\lambda^{-1})}{\lambda^{k+1}}d\lambda \quad (k=0, \pm 1, \pm 2, \cdots, i=\sqrt{-1})$$

for the positively oriented circle $|\lambda|=1/\rho$; and moreover $\sum_{p\geqq 0} c_p\lambda^p$, with domain $\{\lambda: |\lambda|<\infty\}$, associated with the first member on the right of (35) expresses the ordinary part $\tilde{R}(\lambda)$ of $\tilde{T}(1/\lambda)$ and is a finite or an infinite series according as $\tilde{R}(\lambda)$ is a polynomial in $\lambda$ or a transcendental integral function, while $\sum_{p=1}^{\infty} c_{-p}\lambda^{-p}$, with domain $\{\lambda: 1/\rho\leqq|\lambda|\}$, associated with the second member on the right of (35) expresses the sum of the two principal parts of $\tilde{T}(1/\lambda)$ and is essentially an infinite series.

Proof. Let $T(\lambda)=\tilde{T}(1/\lambda)$, and let $D_1, D_2, \cdots, D_{n-1}$, and $D_n$ be the images of $\Delta_1, \Delta_2, \cdots, \Delta_{n-1}$, and $\Delta_n$ by the transformation $1/\lambda$ respectively. Then, by hypotheses, $\{1/\lambda_\nu\}_{\nu=1,2,3,\dots}$ is a bounded infinite sequence, and moreover its closure and the bounded, connected, and closed sets $D_j$ $(j=1, 2, 3, \cdots, n)$ are mutually disjoint. On the other hand, it is also seen by hypotheses that each $1/\lambda_\nu$ is a pole of $T(\lambda)$ in the sense of the functional analysis, that $T(\lambda)$ is regular in the entire complex $\lambda$-plane with the exception of the union of the point $\lambda=\infty$, $\bigcup_{j=1}^{n} D_j$, and the closure of $\{1/\lambda_\nu\}$, that the point $\lambda=\infty$ is a pole or an isolated essential singularity of $T(\lambda)$, and that $T(\lambda)$ has not any term with isolated essential singularity in the domain $\{\lambda: |\lambda|<\infty\}$. In addition, if we replace $\lambda$ in $\sum_{\alpha=1}^{m}\frac{c_\alpha^{(\nu)}}{(\lambda-\lambda_\nu)^\alpha}$ by $1/\lambda$, it is easily verified by direct computation that this sum is taken into

$$\begin{aligned}
\sum_{\alpha=1}^{m}\frac{c_\alpha^{(\nu)}}{(\lambda^{-1}-\lambda_\nu)^\alpha}&=\sum_{\alpha=1}^{m}(-1)^\alpha c_\alpha^{(\nu)}\left\{\lambda_\nu^{-1}\left(1+\frac{\lambda_\nu^{-1}}{\lambda-\lambda_\nu^{-1}}\right)\right\}^\alpha\\
&=\sum_{\alpha=1}^{m}(-1)^\alpha c_\alpha^{(\nu)}\lambda_\nu^{-\alpha}\left\{\sum_{\gamma=0}^{\alpha}\binom{\alpha}{\gamma}\left(\frac{\lambda_\nu^{-1}}{\lambda-\lambda_\nu^{-1}}\right)^\gamma\right\} \quad \left(\binom{\alpha}{0}=1\right)\\
&=\sum_{\alpha=1}^{m}(-1)^\alpha c_\alpha^{(\nu)}\lambda_\nu^{-\alpha}+\sum_{\alpha=1}^{m}(-1)^\alpha\binom{\alpha}{1}c_\alpha^{(\nu)}\lambda_\nu^{-\alpha}\cdot\left(\frac{\lambda_\nu^{-1}}{\lambda-\lambda_\nu^{-1}}\right)\\
&\quad+\sum_{\alpha=2}^{m}(-1)^\alpha\binom{\alpha}{2}c_\alpha^{(\nu)}\lambda_\nu^{-\alpha}\cdot\left(\frac{\lambda_\nu^{-1}}{\lambda-\lambda_\nu^{-1}}\right)^2
\end{aligned}$$

$$+\sum_{\alpha=3}^{m}(-1)^{\alpha}\binom{\alpha}{3}c_{\alpha}^{(\nu)}\lambda_{\nu}^{-\alpha}\cdot\left(\frac{\lambda_{\nu}^{-1}}{\lambda-\lambda_{\nu}^{-1}}\right)^{3}$$
$$+\cdots+(-1)^{m}c_{m}^{(\nu)}\lambda_{\nu}^{-m}\left(\frac{\lambda_{\nu}^{-1}}{\lambda-\lambda_{\nu}^{-1}}\right)^{m},$$

where, by the hypotheses concerning $c_{\alpha}^{(\nu)}$ and $\{\lambda_{\nu}\}$,

$$\sum_{\nu=1}^{\infty}\left\{\sum_{\alpha=r}^{m}(-1)^{\alpha}\binom{\alpha}{r}c_{\alpha}^{(\nu)}\lambda_{\nu}^{-\alpha}\cdot\left(\frac{\lambda_{\nu}^{-1}}{\lambda-\lambda_{\nu}^{-1}}\right)^{r}\right\}$$

is absolutely convergent in the entire complex $\lambda$-plane except for all the points belonging to the closure of $\{1/\lambda_{\nu}\}$ and moreover $\sum_{\nu=1}^{\infty}\sum_{\alpha=1}^{m}|c_{\alpha}^{(\nu)}\lambda_{\nu}^{-\alpha}|<\infty$. These results imply that the principal part of $T(\lambda)$ at each pole $1/\lambda_{\nu}$ in the sense of the functional analysis is expressed in the form $\sum_{\alpha=1}^{m}\frac{d_{\alpha}^{(\nu)}}{(\lambda-\lambda_{\nu}^{-1})^{\alpha}}$ where $\sum_{\nu=1}^{\infty}|d_{\alpha}^{(\nu)}|$ is convergent for every $\alpha=1,2,3,\cdots,m$. Consequently $T(\lambda)$ consists of its ordinary part, its first principal part, and its second principal part in the sense stated in the earlier discussion; and it is clear that the ordinary part of $T(\lambda)$ is a polynomial in $\lambda$ or a transcendental integral function according as zero is a pole or an isolated essential singularity of $\tilde{T}(\lambda)$, that the first principal part of $T(\lambda)$ is given by $\sum_{\nu=1}^{\infty}\sum_{\alpha=1}^{m}\frac{d_{\alpha}^{(\nu)}}{(\lambda-\lambda_{\nu}^{-1})^{\alpha}}$, and that the set of all singularities of $T(\lambda)$ determining completely the second principal part is given by the union of $\bigcup_{j=1}^{n}D_j$ and the set of all those accumulation points of $\{1/\lambda_{\nu}\}$ which do not belong to $\{1/\lambda_{\nu}\}$ itself. Theorem 38 thus leads us to the result that

$$\tilde{T}\left(\frac{1}{\lambda}\right)=T(\lambda)=\sum_{p\geqq 0}c_p\lambda^p+\sum_{p=1}^{\infty}c_{-p}\lambda^{-p}\quad(1/\rho\leqq|\lambda|<\infty),$$

where

$$c_k=\frac{1}{2\pi i}\int_{|\lambda|=1/\rho}\frac{T(\lambda)}{\lambda^{k+1}}d\lambda\quad(k=0,\pm 1,\pm 2,\cdots);$$

and $\sum_{p\geqq 0}c_p\lambda^p$ with domain $\{\lambda\colon |\lambda|<\infty\}$ expresses the ordinary part of $T(\lambda)$ and is a finite or an infinite series according as the point $\lambda=\infty$ is a pole or an isolated essential singularity of $T(\lambda)$, while $\sum_{p=1}^{\infty}c_{-p}\lambda^{-p}$ expresses the sum of the two principal parts of $T(\lambda)$ in the domain $\{\lambda\colon 1/\rho\leqq|\lambda|\}$ and hence is surely an infinite series. Accordingly it is at once obvious that the present theorem is an immediate consequence of these results.

The proof of the theorem is thus complete.