135. Notes on (m, n)-Ideals. II

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The first part of this paper is [2].

Now we give a characterization of groups by means of (m, n)-ideals.

Theorem 4. *A semigroup is a group if and only if it contains no proper (m, n)-ideal, where m, n are arbitrary positive integers.*

Proof. It is evident, that a group contains no proper (m, n)-ideal. Conversely, let us suppose, that the semigroup S contains no proper (m, n)-ideal. Let a be an arbitrary element of S. Then by Corollary of Theorem 2 the products aS and Sa are (m, n)-ideals of S. Hence it follows that $aS=S=Sa$. This means that for every a and b of S there exist solutions x and y in S of the equations

$$ax=b \quad \text{and} \quad ya=b,$$

that is, S is a group.

Corollary. *A semigroup is a group if and only if it contains no proper bi-ideal.*

This is the $m=n=1$ case of Theorem 4, and it is known, see [1], p. 84. (The bi-ideal is same as $(1, 1)$-ideal.)

Theorem 5. *Let m, n are arbitrary positive integers, let S be a semigroup, A be an $(m, 0)$-ideal, B a $(0, n)$-ideal of S, and suppose, that $AB=BA$. Then the product AB is an (m, n)-ideal of S.*

Proof. The suppositions of the theorem imply

$$(AB)(AB)=A^2B^2\subseteq AB,$$

that is, the product AB is a subsemigroup of S. On the other hand

$$(AB)^m S(AB)^n = A^m(B^m SA^n)B^n \subseteq (A^m S)B^n \subseteq AB,$$

i.e. the product AB is an (m, n)-ideal of S.

In the particular case of $m=n=1$, the condition $AB=BA$ is superfluous, that is, we have the following result.

Theorem 6. *Let S be an arbitrary semigroup. If L is a left ideal and R is a right ideal of S, then the product RL is a bi-ideal of S.*

Proof. Since

$$(RL)(RL)\subseteq RL,$$

the product RL is a subsemigroup of S. On the other hand

$$(RL)S(RL)\subseteq RSL\subseteq RL,$$

that is, the product RL is a bi-ideal of S, as we stated.

If S is a regular semigroup, that is, $a\in aSa$ for each element a

in S, then the converse of Theorem 6 also holds.

Theorem 7. *A subset A of a regular semigroup s is a bi-ideal of S if and only if S contains a left ideal L and a right ideal R, such that*

$$A=RL.$$

Proof. We prove that if S is a regular semigroup, and A is a bi-ideal of S, then

$$A=(A\cup AS)(A\cup SA).$$

First, we see that

$$A\subseteq(A\cup AS)(A\cup SA)=A^2\cup ASA,$$

because of $a=axa\in ASA$ for each a in A. Conversely,

$$(A\cup AS)(A\cup SA)\subseteq A,$$

because of A is a bi-ideal of S, i.e. $A^2\subseteq A$ and $ASA\subseteq A$. Thus in view of Theorem 6, Theorem 7 is proved.

A more general result as that of Theorem 6 is contained in the following theorem.

Theorem 8. *The product of a bi-ideal and of a non-empty subset of a semigroup S is also a bi-ideal of S.*

Proof. Let S be a semigroup, A be a non-empty subset, and B a bi-ideal of S. Then

$$(AB)(AB)\subseteq AB,$$

that is, the product AB is a subsemigroup of S. On the other hand

$$(AB)S(AB)\subseteq A\cdot BSB\subseteq AB,$$

which shows that AB is a bi-ideal of the semigroup S.

Analogously we can prove that the product BA is also a bi-ideal of the semigroup S.

The result of Theorem 8 was recently proved in author's paper [3].

Theorem 9. *Let S be a semigroup, A be an (m,n)-ideal of S, and B be an (m,n)-ideal of the semigroup A such that $B^2=B$. Then B is an (m,n)-ideal of S.*

Proof. It is trivial that B is a subsemigroup of S. Secondly, since

$$A^mSA^n\subseteq A, \quad \text{and} \quad B^mAB^n\subseteq B,$$

we have

$$B^mSB^n=B^m(B^mSB^n)B^n\subseteq B^m(A^mSA^n)B^n\subseteq B^mAB^n\subseteq B,$$

therefore B is an (m,n)-ideal of S.

References

[1] A. H. Clifford and G. B. Preston: The Algebraic Theory of Semigroups, vol. I. Providence (1961).

[2] S. Lajos: Notes on (m,n)-ideals. I. Proc. Japan Acad., **39**, 419–421 (1963).

[3] ——: On ideal theory for semigroups (in Hungarian). A Magyar Tud. Akad. Mat. Fiz. Oszt. Közl., **11**, 57–66 (1961).