159. *Remarks on Ninomiya's Domination Principle*

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1. Introduction. In the n-dimensional Euclidean space R^n $(n\geq 1)$, the potential of a given order α, $0<\alpha<n$, of a measure μ in R^n is defined by

$$U_\alpha^\mu(x)=\int |x-y|^{\alpha-n}d\mu(y),$$

provided the integral on the right exists. The kernel $|x-y|^{\alpha-n}$ will be called the kernel of order α. Let μ be a measure in R^n. When the integral

$$\iint |x-y|^{\alpha-n}d\mu(y)d\mu(x)$$

exists, we shall call it the *α-energy* of μ. We shall denote the inner capacity of a set A with respect to the kernel of order α by $C_\alpha(A)$. A property is said to hold *α-p.p.p.* on a subset X in R^n, when the property holds on X except a set E with $C_\alpha(E)=0$. The measure μ in R^n will be said to be *α-finite*, when the potential $U_\alpha^\mu(x)$ is defined and finite *α-p.p.p.* in R^n. We shall denote the support of a measure μ in R^n by S_μ.

Ninomiya [3] proved the following domination principle.

In R^n $(n\geq 3)$, let α be a positive number such that $0<\alpha\leq 2$, let μ be a positive measure with compact support such that the α-energy is finite, and let ν be a positive measure in R^n. If

$$U_\alpha^\mu(x)\leq U_\alpha^\nu(x) \qquad \text{on } S_\mu,$$

then

$$U_\beta^\mu(x)\leq U_\beta^\nu(x) \qquad \text{in } R^n$$

for any β such that $\alpha\leq\beta<n$.

He proved that the same domination principle is valid in R^2 if $0<\alpha\leq 1$. In this paper, we shall prove Ninomiya's domination principle in a possibly general form.

2. Ninomiya's domination principle. Lemma.[1] *In R^n $(n\geq 1)$, let α be a positive number such that $0<\alpha\leq 2$, $0<\alpha<2$ or $0<\alpha<1$ according to $n\geq 3$, $n=2$ or $n=1$. Then the kernel of order α satisfies the balayage principle with respect to the kernel of order β for any β such that $\alpha\leq\beta<n$, namely, for any p in R^n and any closed set F, there exists a positive measure λ, supported by F, such that*

$$U_\alpha^\lambda(x)=|x-p|^{\beta-n} \qquad \alpha\text{-p.p.p. on } F,$$

1) Ninomiya (Theorem 2 in [3]) showed this when $n\geq 3$ and F is compact.

$$U_\alpha^\lambda(x) \leq |x-p|^{\beta-n} \quad in\ R^n.$$

Proof. To avoid the trivial case, we assume that $C_\alpha(F)>0$ and $\alpha<\beta$. By the formula of Riesz,[2]

$$\int |x-y|^{\alpha-n}|y-p|^{(\beta-\alpha)-n}dy = K_{\alpha,\beta-\alpha}|x-p|^{\beta-n}$$

where

$$K_{\alpha,\beta-\alpha} = \pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta-\alpha}{2}\right)\Gamma\left(\frac{n-\beta}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-\beta+\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}.$$

Consider the following positive measure

$$\frac{1}{K_{\alpha,\beta-\alpha}}|y-p|^{(\beta-\alpha)-n}dy.$$

This is α-finite, so that by the theorem of Riesz,[3] there exists its balayaged measure λ to F. Consequently

$$U_\alpha^\lambda(x) = |x-p|^{\beta-n} \quad \alpha\text{-}p.p.p. \text{ on } F,$$
$$U_\alpha^\lambda(x) \leq |x-p|^{\beta-n} \quad \text{in } R^n.$$

By this lemma, we obtain the following

Theorem 1.[4] *In R^n $(n\geq 1)$, let α be a positive number such that $0<\alpha\leq 2$, $0<\alpha<2$ or $0<\alpha<1$ according to $n\geq 3$, $n=2$ or $n=1$, let μ be a positive measure such that the α-energy is finite, and let ν be a positive measure in R^n. If*

$$U_\alpha^\mu(x) \leq U_\alpha^\nu(x) \quad on\ S_\mu,$$

then

$$U_\beta^\mu(x) \leq U_\beta^\nu(x) \quad in\ R^n$$

for any β such that $\alpha\leq\beta<n$.

Proof. Let p be a point in R^n. By the above lemma, there exists a positive measure λ, supported by S_μ, such that

$$U_\alpha^\lambda(x) = |x-p|^{\beta-n} \quad \alpha\text{-}p.p.p. \text{ on } S_\mu,$$
$$U_\alpha^\lambda(x) \leq |x-p|^{\beta-n} \quad \text{in } R^n.$$

Since the α-energy of μ is finite, $\mu(A)=0$ for any set A with $C_\alpha(A)=0$. Hence

$$\begin{aligned} U_\beta^\mu(p) &= \int |x-p|^{\beta-n}d\mu(x) = \int U_\alpha^\lambda(x)d\mu(x) \\ &= \int U_\alpha^\mu(x)d\lambda(x) \leq \int U_\alpha^\nu(x)d\lambda(x) \\ &= \int U_\alpha^\lambda(x)d\nu(x) \leq \int |x-p|^{\beta-n}d\nu(x) \\ &= U_\beta^\nu(p). \end{aligned}$$

This completes the proof.

2) Cf. [1], p. 151.

3) Cf. [4], pp. 21-22.

4) Ninomiya [3] proved this in the case that $n\geq 3$ and S_μ is compact.

Another form of Ninomiya's domination principle is the following

Theorem 2.[5] *In R^n $(n\geq 1)$, let α be a positive number such that $0<\alpha\leq 2$, $0<\alpha<2$, and $0<\alpha<1$ according to $n\geq 3$, $n=2$ or $n=1$, and let β be a positive number such that $\alpha\leq\beta<n$. Let μ be a positive measure such that the α-energy is finite, and let ν be a positive measure in R^n. If*

$$U_\alpha^\mu(x)\leq U_\beta^\nu(x) \qquad \text{on } S_\mu,$$

then

$$U_\alpha^\mu(x)\leq U_\beta^\nu(x) \qquad \text{in } R^n.$$

Proof. By virtue of Theorem 1 we may assume that $\alpha<\beta$. Let p be a point in CS_μ. By the above lemma, there exists a positive measure λ such that

$$U_\alpha^\lambda(x)=|x-p|^{\alpha-n} \qquad \alpha\text{-p.p.p. on } S_\mu,$$
$$U_\alpha^\lambda(x)\leq|x-p|^{\alpha-n} \qquad \text{in } R^n.$$

Then

$$U_\alpha^\mu(p)=\int U_\alpha^\lambda(x)d\mu(x)=\int U_\alpha^\mu(x)d\lambda(x)\leq\int U_\beta^\nu(x)d\lambda(x)$$
$$=\int U_\beta^\lambda(x)d\nu(x).$$

On the other hand

$$\begin{aligned}U_\beta^\lambda(x)&=\int|x-y|^{\beta-n}d\lambda(y)\\&=\frac{1}{K_{\alpha,\beta-\alpha}}\iint|x-z|^{(\beta-\alpha)-n}|y-z|^{\alpha-n}dz\,d\lambda(y)\\&=\frac{1}{K_{\alpha,\beta-\alpha}}\iint|x-z|^{(\beta-\alpha)-n}|y-z|^{\alpha-n}d\lambda(y)dz\\&\leq\frac{1}{K_{\alpha,\beta-\alpha}}\int|x-z|^{(\beta-\alpha)-n}|p-z|^{\alpha-n}dz\\&=|x-p|^{\beta-n}.\end{aligned}$$

Consequently

$$\int U_\beta^\lambda(x)d\nu(x)\leq\int|x-p|^{\beta-n}d\nu(x)=U_\beta^\nu(p).$$

This completes the proof.

By using Theorem 2 and Ninomiya's general theory [2], we obtain the following

Proposition.[6] *In R^n $(n\geq 1)$, let α be a positive number such that $0<\alpha\leq 2$, $0<\alpha<2$ or $0<\alpha<1$ according to $n\geq 3$, $n=2$ or $n=1$, and let β be a positive number such that $\alpha\leq\beta<n$. Then for any β-finite positive measure μ and any closed set F, there exists a positive measure μ', supported by F, such that*

$$U_\alpha^{\mu'}(x)=U_\beta^\mu(x) \qquad \alpha\text{-p.p.p. on } F,$$
$$U_\alpha^{\mu'}(x)\leq U_\beta^\mu(x) \qquad \text{in } R^n.$$

5) Ninomiya [3] proved this in the case that $n\geq 3$ and S_μ is compact.

6) Ninomiya [3] proved this in the case that $n\geq 3$ and F is compact.

References

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