158. On the Spectra of Uniformly Increasing Mappings

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Let E be a real Banach space, G be an open set and $\bar{G}$ be its closure.

In [2], we have given the following definition:

A mapping f of $\bar{G}$ in E is said to be $(\varepsilon_0, \delta_0)$-uniformly increasing at $a \in G$ if

(i) $a+x \in \bar{G}$ *if* $||x|| \leqq \delta_0$;

(ii) $||f_a(x)-\alpha x|| \geqq \varepsilon_0 ||x||$ *for any non-positive number α and any element x such that $||x|| \leqq \delta_0$, where* $f_a(x)=f(a+x)-f(a)$.

The purpose of this paper is to prove the following

Theorem. *Assume that*

1. $F(x)$ *is a completely continuous mapping of $\bar{G}$ in E;*
2. $F(a)=\lambda_0 a$ *for some* $\lambda_0 \neq 0$ *and some* $a \in G$;
3. $f(x)=x-\frac{1}{\lambda_0}F(x)$ *is $(\varepsilon_0, \delta_0)$-uniformly increasing at a.*

Then, we have that

1°. *a is an isolated fixed point of* $\frac{1}{\lambda_0}F(x)$;

2°. *For any λ such that* $|\lambda-\lambda_0| < \min.\left\{|\lambda_0|, \frac{|\lambda_0|\varepsilon_0\delta_0}{||a||+\delta_0}\right\}$, *there exists x_λ such that*

$$F(x_\lambda)=\lambda x_\lambda \quad and \quad ||x_\lambda - a|| \leqq \frac{1}{|\lambda_0|\varepsilon_0}(||a||+\delta_0)|\lambda-\lambda_0|.$$

Remark. A mapping $F(x)$ is said to be *completely continuous on* $\bar{G}$ if it is continuous and the image $F(\bar{G})$ is contained in a compact set.

Proof. 1°. Assume that a is not an isolated fixed point of $\frac{1}{\lambda_0}F(x)$, then there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} x_n = 0 \quad \text{and} \quad F(a+x_n)=\lambda_0(a+x_n).$$

Since $f(x)=x-\frac{1}{\lambda_0}F(x)$, we have

$$f(a+x_n)=(a+x_n)-\frac{1}{\lambda_0}F(a+x_n)=0.$$

Now, since $f(x)$ is $(\varepsilon_0, \delta_0)$-uniformly increasing at a, we have

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$$\varepsilon_0\|x_n\| \leqq \left\| f_a(x_n) - \left(-\frac{1}{2}\varepsilon_0\right)x_n \right\| = \left\| f(a+x_n) - f(a) + \frac{1}{2}\varepsilon_0 x_n \right\| = \frac{1}{2}\varepsilon_0\|x_n\|$$

for such large n that $\|x_n\| \leqq \delta_0$. This is a contradiction.

2°. Let us take λ which satisfies our condition and consider the completely continuous mapping $\frac{1}{\lambda}F(x)$ of $\bar{G}$ in E. To this mapping we shall apply a fixed point theorem which we have proved in [1] (see the *remark* below). Now, assume that

$$\frac{1}{\lambda}F(x) = \alpha x + (1-\alpha)a$$

for some number α and some element x such that $\|x-a\| = \delta_0$. Then, we have

$$\begin{aligned} f_a(x-a) &= f(x) - f(a) = f(x) \\ &= x - \frac{1}{\lambda_0}F(x) \\ &= \frac{\lambda}{\lambda_0}(1-\alpha)(x-a) + \left(1 - \frac{\lambda}{\lambda_0}\right)x, \end{aligned}$$

namely,

$$\begin{aligned} &\left\| f_a(x-a) - \frac{\lambda}{\lambda_0}(1-\alpha)(x-a) \right\| \\ &= \left|1 - \frac{\lambda}{\lambda_0}\right| \|x\| \leqq \frac{1}{|\lambda_0|}|\lambda_0 - \lambda| \frac{\|x\|}{\|x-a\|}\|x-a\| \\ &\leqq \frac{1}{|\lambda_0|}|\lambda_0-\lambda| \frac{\|a\|+\delta_0}{\delta_0}\|x-a\| < \varepsilon_0\|x-a\|, \end{aligned}$$

which implies that $\frac{\lambda}{\lambda_0}(1-\alpha) > 0$. Since $\frac{\lambda}{\lambda_0} > 0$, we have $\alpha < 1$. Therefore, there exists x_λ such that

$$\|x_\lambda - a\| \leqq \delta_0 \quad \text{and} \quad F(x_\lambda) = \lambda x_\lambda.$$

Finally, we prove the following inequality

$$\|x_\lambda - a\| \leqq \frac{1}{\varepsilon_0|\lambda_0|}(\|a\| + \delta_0)|\lambda - \lambda_0|.$$

For this purpose, we need the following fact which can be easily derived from the definition of the uniform increasingness: *Let f be $(\varepsilon_0, \delta_0)$-uniformly increasing at a. Then*

$$\|f_a(x) - \alpha x\| \geqq \frac{\|a\|}{\|a\|+\delta_0}\varepsilon_0\|x\|$$

for any number α and element x such that

$$\alpha \leqq \frac{\delta_0}{\|a\|+\delta_0}\varepsilon_0 \quad \textit{and} \quad \|x\| \leqq \delta_0.$$

Using this fact, since $\|x_\lambda - a\| \leqq \delta_0$ and $1 - \frac{\lambda}{\lambda_0} \leqq \frac{\delta_0}{\|a\|+\delta_0}\varepsilon_0$, we have

$$\frac{\|a\|}{\|a\|+\delta_0}\varepsilon_0\|x_\lambda - a\| \leqq \left\| f_a(x_\lambda - a) - \left(1 - \frac{\lambda}{\lambda_0}\right)(x_\lambda - a) \right\|$$

$$=\left\|f(x_\lambda)-\left(1-\frac{\lambda}{\lambda_0}\right)(x_\lambda-a)\right\|$$
$$=\left\|x_\lambda-\frac{1}{\lambda_0}F(x_\lambda)-\left(1-\frac{\lambda}{\lambda_0}\right)(x_\lambda-a)\right\|$$
$$=\left|1-\frac{\lambda}{\lambda_0}\right|\cdot\|a\|$$
$$=\frac{1}{|\lambda_0|}|\lambda_0-\lambda|\cdot\|a\|,$$

from which our inequality follows.

Remark 1. In [1], we have proved the following fixed point theorem which we used in the above proof: *Let E be a locally convex topological linear space over the real number field. Let G be an open set, $\bar{G}$ be its closure and ∂G be its boundary. Let F be a completely continuous mapping of $\bar{G}$ in E. Then, the mapping F has a fixed point in $\bar{G}$ if there exists an element $a\in G$ such that the following condition is satisfied:*

"*if $F(x)=\alpha x+(1+\alpha)a$ for some $x\in\partial G$ and some number α then $\alpha\leqq 1$*".

Remark 2. Under the assumptions of our theorem, if the mapping F is Fréchet-differentiable at a, then λ_0 is not the proper value of the Fréchet-derivative of F at a. Therefore, in this case, our theorem is contained in the so-called Implicit Function Theorem.

References

[1] S. Yamamuro: Some fixed point theorems in locally convex linear spaces. Yokohama Math. Journ., **11**, 5–12 (1963).

[2] ——: Monotone mappings in topological linear spaces. Journ. Australian Math. Soc., to appear.