148. *Differential Forms of the Second Kind on Algebraic Varieties with Certain Imperfect Ground Fields*[*)]

By Satoshi ARIMA

Department of Mathematics, Musashi Institute of Technology, Tokyo

(Comm. by Zyoiti SUETUNA, M.J.A., Nov. 12, 1964)

1. Let V be an irreducible projective variety of dimension r defined over a field k of characteristic $p \neq 0$. We assume that the field K of rational functions of V over k is a regular extension of k. A subvariety W of V/k will mean a subvariety which is defined and irreducible over k; the local ring of W on V/k will be denoted by $\mathfrak{o}_W$, and the integral closure of $\mathfrak{o}_W$ in K by $\bar{\mathfrak{o}}_W$. A subvariety W of V/k is simple (over k) if and only if the local ring $\mathfrak{o}_W$ is regular.

We shall consider derivations and differential forms of K *over* k^p or, equivalently, over K^p. (They need not necessarily be trivial on k.) The exterior differential $d\omega$ of a differential form ω of K is therefore to be understood also in that sense. We have $d(z^p\omega)=z^p d\omega$, $z \in K$. Following the usual definition (in [3]) we define holomorphic differential forms and derivations as follows. A derivation ∂ of K is *holomorphic at a subvariety* W of V/k if $\partial\bar{\mathfrak{o}}_W \subseteqq \bar{\mathfrak{o}}_W$. A differential form ω of K of degree q is *holomorphic at* W if $\omega(\partial_1, \cdots, \partial_q) \in \bar{\mathfrak{o}}_W$ for all derivations $\partial_1, \cdots, \partial_q$ which are holomorphic at W. We denote by $\mathcal{D}_1(V)$ the k-vector space of all differential forms of K of degree 1 which are holomorphic at every subvariety of V/k. A differential form ω of K of degree q is *of the second kind at a subvariety* W if $\omega - d\theta_W$ is holomorphic at W with a suitable differential form θ_W of degree $q-1$. If ω is of the second kind at W, then $a^p\omega$ with $a \in k$ is also of the second kind at W. In fact, if $\omega - d\theta$ is holomorphic at W, then $a^p\omega - d(a^p\theta) = a^p(\omega - d\theta)$ is holomorphic at W since $a^p \in \bar{\mathfrak{o}}_W$. All *closed* differential forms of K of degree 1 which are of the second kind at every subvariety of V/k form therefore a k^p-vector space $\mathcal{D}_2(V)$, and the set $\mathcal{D}_e(V)$ of differentials dz ($z \in K$) is a k^p-vector subspace of $\mathcal{D}_2(V)$. The purpose of the present note is to show that the dimension over k^p of the factor space $\mathcal{D}_2(V)/\mathcal{D}_e(V)$ equals the dimension over k of the space $\mathcal{D}_1(V)$ *under the assumption* $[k:k^p] < \infty$. In the case where k is perfect, our result implies the known equality $\dim_k \mathcal{D}_2(V)/\mathcal{D}_e(V) = \dim_k \mathcal{D}_1(V)$, which we have shown in [1]

*) After the completion of this paper, the auther found that the same result had been obtained in a more general form by E. Kunz in "Einige Anwendungen des Cartier-Operators", Arch. d. Math., **13**, 349-356 (1962). However, our treatment, using derivations and *uniformization* of relatively simple points introduced by Zariski-Falb [4], is essentially different from that of Kunz.

with the universal domain k. (In [1], k was assumed to be the universal domain, but it is easy to see that k needs only to be perfect; then all the arguments of [1] remain valid.)

2. We assume that $[k:k^p]<\infty$ and put $[k:k^p]=p^\rho$. We first recall the definition and results on uniformizing coordinates in [4] under our assumption.

The elements $\xi_1,\cdots,\xi_{\rho+r}\in K$ are *uniformizing coordinates* of a subvariety W if $\xi_1,\cdots,\xi_{\rho+r}\in \mathfrak{o}_W$ and there exist $\rho+r$ derivations $\partial_i=\dfrac{\partial}{\partial\xi_i}$ of K, holomorphic at W, such that $\partial_i\xi_j=\delta_{ij}$ $(1\leqq i,\ j\leqq\rho+r)$. If W is simple on V/k, then there exist uniformizing coordinates, say, $\xi_1,\cdots,\xi_{\rho+r}$ of W; every derivation ∂ of K is written uniquely as $\partial=\sum_{1\leqq i\leqq\rho+r} z_i\partial_i$ with $z_i\in K$, and ∂ is holomorphic at W if and only if $z_i\in\mathfrak{o}_W$ $(1\leqq i\leqq\rho+r)$; every differential form ω of K of degree q is written uniquely as $\omega=\sum_{1\leqq i_1<\cdots<i_q\leqq\rho+r} z_{i_1\cdots i_q}d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_q}$ with $z_{i_1\cdots i_q}\in K$, and ω is holomorphic at W if and only if $z_{i_1\cdots i_q}\in\mathfrak{o}_W$ $(1\leqq i_1<\cdots<i_q\leqq \rho+r)$; moreover it can be seen that $K=K^p(\xi_1,\cdots,\xi_{\rho+r})$ and $\{\xi_1,\cdots, \xi_{\rho+r}\}$ is a p-basis of K/K^p.

The following two propositions are generalizations of the results in [3]; the details of the proof will be omitted since the proof in [3] can be transferred almost literally to our cases. A normal subvariety W of V/k of codimension 1 gives rise to a discrete valuation of rank 1 of the field K over k which will be denoted by v_W.

PROPOSITION 1. *Assume that the variety V/k is normal. Let ω be a differential form of K of degree $\rho+r$, $\omega\neq 0$. Then we have $v_W(\omega)=0$ for all but a finite number of subvarieties W of codimension* 1.

To prove this proposition, we may replace the variety V by its affine representative V_a. Let $k[x_1,\cdots,x_N]$ be the coordinate ring of V_a with a separating transcendence basis $\{x_1,\cdots,x_r\}$ of K/k, and let $\{u_1,\cdots,u_\rho\}$ be a p-basis of k/k^p. Then it will be easily shown that $u_1,\cdots,u_\rho,x_1,\cdots,x_r$ are uniformizing coordinates of all but a finite number of subvarieties W of codimension 1 of V/k. Let $\omega=Adu_1\wedge\cdots\wedge du_\rho\wedge dx_1\wedge\cdots\wedge dx_r$ with $A\in K$. Then $v_W(\omega)=v_W(A)$ for all but a finite number of subvarieties W of codimension 1, which proves Proposition 1.

PROPOSITION 2. *Assume that the variety V/k is normal. Then we have* $\dim_k\mathcal{D}_1(V)<\infty$.

To prove this, let $\{\xi_1,\cdots,\xi_{\rho+r}\}$ be a p-basis of K/K^p. Let $\mathfrak{a}$ be the set of subvarieties W of codimension 1 such that $\xi_1,\cdots,\xi_{\rho+r}$ are not uniformizing coordinates of W. With the help of Proposition 1, it will be seen that $\mathfrak{a}$ is a finite set. Let $\mathfrak{a}=\{W_1,\cdots,W_h\}$. Then we can find integers $s_i=s(W_i)\leqq 0$ $(1\leqq i\leqq h)$ such that $(z_i)+Z\geqq 0$ for every

$\omega = z_1 d\xi_1 + \cdots + z_{\rho+r} d\xi_{\rho+r} \in \mathcal{D}_1(V)$, where $Z = -\sum_{1 \leq i \leq h} s_i W_i$. The k-linear mapping $\mathcal{D}_1(V_k) \ni \omega = \sum_{1 \leq i \leq \rho+r} z_i d\xi_i \to (z_1, \cdots, z_{\rho+r}) \in \prod L(Z)$ is injective, where $L(Z) = \{z \in K \mid (z) + Z \geqq 0\}$. $L(Z)$ is of finite dimension over k, since K/k is a regular extension. (Cf. [5], p. 357.) It follows therefore that $\dim_k \mathcal{D}_1(V) < \infty$.

3. Let $\{\xi_1, \cdots, \xi_{\rho+r}\}$ be a p-basis of K/K^p. Then a closed differential form ω of K of degree $q \geqq 1$ is written uniquely in the form $\omega = d\theta + \sum_{i_1 < \cdots < i_q} z^p_{i_1 \cdots i_q} \xi^{p-1}_{i_1} d\xi_{i_1} \wedge \cdots \wedge \xi^{p-1}_{i_q} d\xi_{i_q}$ with $z_{i_1 \cdots i_q} \in K$, and the differential form $C\omega$ of K is defined by the formula

(1) $$C\omega = \sum_{i_1 < \cdots < i_q} z_{i_1 \cdots i_q} d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_q}.$$

If ω is of degree 1, (1) is equivalent to the formula

(2) $$(C\omega(\partial))^p = \omega(\partial^p) - \partial^{p-1}(\omega(\partial)) \quad \text{for any derivation } \partial \text{ of } K.$$

C is p^{-1}*-semi-linear* in the sense

$$C(\omega + \omega') = C\omega + C\omega', \quad C(z^p \omega) = zC\omega, \ z \in K;$$

$C\omega = 0$ if and only if ω is exact. (Cf. Cartier [2].)

PROPOSITION 3. *Let ω be a closed differential form of K of degree 1. If ω is of the second kind at a subvariety W of V/k, then $C\omega$ is holomorphic at W.*

PROOF. Let $\theta = \omega - df$ be holomorphic at W. Take any derivation ∂ of K with $\partial \bar{\mathfrak{o}}_W \subseteqq \bar{\mathfrak{o}}_W$; then we have $(C\omega(\partial))^p = \partial(\theta^p) - \partial^{p-1}(\theta(\partial)) \in \bar{\mathfrak{o}}_W$ by (2), since θ is holomorphic at W. Since $\bar{\mathfrak{o}}_W$ is integrally closed in K and since $C\theta(\partial) \in K$, we have $C\theta(\partial) \in \bar{\mathfrak{o}}_W$. Thus $C\omega = C\theta$ is holomorphic at W.

PROPOSITION 4. *Let ω be a closed differential form of degree $q \geqq 1$ of K. If $C\omega$ is holomorphic at a simple subvariety W of V/k, then ω is of the second kind at W.*

PROOF. Let $\{\xi_1, \cdots, \xi_{\rho+r}\}$ be a set of uniformizing coordinates of W on V/k. Then, since $\{\xi_1, \cdots, \xi_{\rho+r}\}$ is a p-basis of K/K^p, ω is of the form $\omega = d\theta + \sum_{i_1 < \cdots < i_q} z^p_{i_1 \cdots i_q} \xi^{p-1}_{i_1} d\xi_{i_1} \wedge \cdots \wedge \xi^{p-1}_{i_q} d\xi_{i_q}$, and we have $C\omega = \sum_{i_1 < \cdots < i_q} z_{i_1 \cdots i_q} d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_q}$. If $C\omega$ is holomorphic at W, then we have $z_{i_1 \cdots i_q} \in \mathfrak{o}_W$ and so $z^p_{i_1 \cdots i_q} \xi^{p-1}_{i_1} \cdots \xi^{p-1}_{i_q} \in \mathfrak{o}_W$, which implies that $\omega - d\theta$ is holomorphic at W. This completes the proof.

From Propositions 1 and 2 follows immediately

THEOREM. *Let V be an irreducible projective variety defined over a field k of prime characteristic p, and assume that the field of rational functions on V over k is a regular extension of k. If V/k is non-singular and $[k : k^p] < \infty$, then $\omega \to C\omega$ induces a p^{-1}-semi-linear bijective mapping of the k^p-space $\mathcal{D}_2(V)/\mathcal{D}_e(V)$ onto the k-space $\mathcal{D}_1(V)$; especially we have* $\dim_{k^p} \mathcal{D}_2(V)/\mathcal{D}_e(V) = \dim_k \mathcal{D}_1(V)$.

References

[1] S. Arima: Differential forms of the first and second kind on modular algebraic varieties. J. Math. Soc. Japan, **16**, 102–108 (1964).
[2] P. Cartier: Questions de rationalité des diviseurs en géométrie algébrique. Bull. Soc. Math. France, **86**, 177–251 (1958).
[3] O. Zariski: An introduction to the theory of algebraic surfaces (lecture notes). Harvard University (1957–58).
[4] O. Zariski and P. Falb: On differentials in function fields. Amer. J. Math., **88**, 542–556 (1961).
[5] O. Zariski and P. Samuel: Commutative Algebra, vol. II. Princeton, D. Van Nostrand Co. (1960).