177. The Riemann Lebesgue's Theorem and its Application to Cut-off Process

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§1. **Introduction.** In order to avoid the divergence in perturbation for Miyatake Van-Hove model with fixed point source [4–6] various methods are deviced. The most usual and important method of them is the cut-off operation whose true meaning is to use a source with non zero volume (in some meaning). But in cut-off process the causality condition is not satisfied. In quantum field theory, the cut-off operation is used without any hesitation.

Now, let's rewrite this operation to the mathematical form. The periodic function $f(x)$ with period 2π (by considering the physical effect to enclose in a box) can be developed to the series $f(x)=\sum_{n=-\infty}^{\infty} C_n e^{inx}$. Well known Riemann-Lebesgue's theorem is the following

Theorem. *If $f(x)$ is contained in L^1, then* $\lim_{|n|\to\infty} C_n=0$.

Our requirement is the very troublesome one. Namely, it is desired that the above theorem (by the order $o(1/n)$) is satisfied for $f(x)$ whose definition's domain is the set of isolated points. The function $f(x)$ used in Miyatake Van-Hove model was $\delta(x)$ defined in the interval $[0, 2\pi]$. Afterward, O. Miyatake has used the function $f(x)=\sum_{i=1}^{n} C_i\delta(x_i)$ defined in the interval $[0, 2\pi]$ and has investigated whether this requirement is satisfied or not for this $f(x)$. But the above requirement is not satisfied for these models. Here, as $f(x)$, we will use the finite linear combination of the characteristic functions of nowhere dense perfect set with positive measure appearing in the process tending to δ function or δ-like function (instead of the sum of δ function defined in the set of isolated points), and we will investigate whether the above requirement is satisfied or not. Because, "nowhere dense" corresponds to "isolated" by some meaning, and "perfect" corresponds to "related." The cut-off related to only Riemann-Lebesgue's theorem is called "natural cut-off." For the requirement to the order tending to zero ($o(1/n)$) it seems that "exact cut-off" must be used. For the exact cut-off (by using A-integral) even a sort of countable infinite linear combination of the above characteristic functions is used. The foundation of our consideration is the A-integral representation of distributions (or E. R. integral) by Виноградова Бонди etc. [1–2]. The carrier of the representing function $f(x)$ in A-integral representation has the following properties:

(a) it is the countable infinite sum of nowhere dense perfect sets,

(b) the carrier of $[f]_n(x)$ is nowhere dense perfect set for any positive number n,

(c) for any positive $\varepsilon>0$ and $\varphi\in(D)$ (by L. Schwartz) there exists a positive integer N such that for $n>N$

mes [carrier $\{(f(x)-[f]_n(x))\varphi(x)\}]<\varepsilon$ and $\left|\int\{f(x)-[f]_n(x)\}\varphi(x)dx\right|<\varepsilon$.

Here

$$[f]_n(x)=\begin{cases} f(x) & \text{for } x \text{ such that } |f(x)|\leqq n \\ 0 & \text{for } x \text{ such that } |f(x)|>n. \end{cases}$$

Afterwards, this notation is often used. Our true aim is to use the A-integrable function with this carrier instead of smooth functions. In exact cut-off this A-integrable function with the same power as the smooth function is used.

The another proof of the Riemann Lebesgue's theorem showing this circumstance etc., and the difference between exact cut-off and natural cut-off are shown in § 2. It seems that our theory shows the inner structure of the elementary particle which satisfies the causality condition (in the generalized meaning by using Dini's derivative as velocity). In § 3, using this, a model showing the processes tending to various physical models is constructed. This model is useful to search the valuable conditional convergent sequence tending to δ function used in quantum field theory [8].

§ 2. The Riemann Lebesgue's theorem related to the characteristic function of no where dense perfect set. (1) It is well known that the sequence of functions $1/2\pi\int_{-n}^{n}e^{ikx}dk=d_n(x)$ (or other regular functions) converging to $\delta(x)$ (as n tends to ∞) can be constructed. But the carriers of these $d_n(x)$ are the interval $(-\infty, \infty)$. Our purpose is to obtain the functions $f_n(x)([f_n]_m(x))$ such that the carriers of them has the properties described in § 1 and they play the same (or the resemble) role as $d_n(x)$. From this reason the A-integral representation of distribution obtained by Бонди [1], Виноградова [2] etc. is very useful. By their method δ function can be also represented by A-integrable function f_δ whose carrier has the properties described in § 1, and $[f_\delta]_m$ can be used for the resemble purpose. The carrier of the functions $f_n(x)$, $f_\delta(x)$, $[f_n]_m(x)$ and $[f_\delta]_m(x)$ corresponds to the set of the position of the point source.

Definition. If the function $f(x)$ defined in the interval $[a, b]$ satisfy the following conditions:

1) mes $\{x; x\in[a, b], |f(x)|>n\}=o(1/n)$

2) there exists a limit $\lim_{n\to\infty}\int_a^b[f]_n(x)dx$,

then we say that $f(x)$ is A-integrable and the above limit is A-integral of $f(x)$.

A-integral representation in [1–2] is very complicated but it has many advantages.

(2) Next, let's show the concrete construction of nowhere dense perfect set with positive measure. Namely, choose an interval [0, 1] and take off an open interval with measure 1/5 from the middle of the first interval [0, 1]. Next take off two open intervals with measure $(1/5)^2$ from the middle of two residual intervals. By the iteration of the same process countable infinite times no where dense perfect set with positive measure is constructed. Let's show this. (i) The measure of this set is $1-(1/5+2/5^2+4/5^3+\cdots)=1-1/5/(1-2/5)=2/3$. (ii) Let I_n denote the closed set constructed by n th process. Since the relation $[0, 1]=I_0\supset I_1\supset\cdots\supset I_n\supset\cdots$ holds good, then $E=\cap_{n=0}^{\infty} I_n$ is a closed set. For any neighbourhood $U(p)$ of the point p contained in E, there exists a point p'' contained in I_0-E. Since *there is not any inner point contained in the closure of* E, then E is a no where dense set. (iii) For any neighbourhood $U(p)$ of the point p contained in E, there exists a point p' contained in E such that $p\neq p'$. Since E *is closed and it does not contain any isolated point*, then E is a perfect set. The Riemann Lebesgue's theorem is well known [7] and in Fourier analysis this theorem is used to treat the usual functions. But in the consideration of point source, δ function etc. are usually considered. Therefore, let's give here another proof which shows the results with respect to these singular functions.

Let $I_E(x)$ denote the characteristic function of the set E. Namely

$$I_E(x)=\begin{cases}1 & \text{for } x\in E\\ 0 & \text{for } x\in E^c\,.\end{cases}$$

Suppose that E is a set with positive measure contained in the interval $I_0=[0, 1]$.

Lemma 1. *For measurable set* $E\subset[0, 1]$, $\int_0^1 I_E(x)\exp(2n\pi ix)dx=C_n$ *tends to zero by the order* $O(1/n^{1-\varepsilon})$ (*for any* $\varepsilon>0$) *as* n *tends to* ∞.

Proof. $I_E(x)\in L^\infty[0, 1]\subset L^n[0, 1]$ for $1\leqq n<+\infty$. Namely, it is bounded. Then the conclusion of this lemma is obtained. But, for the above constructed set, it seems that the order of C_n tending to zero is not $o(1/n)$ ([7], p. 19). Hence the exact cut-off must be used.

Theorem 1. *Suppose that for any* $\varepsilon>0$, *there exist a set of finite disjoint intervals* $\tilde{I}_k(k=1, 2, \cdots, M)$ (M *is depending to this set*) *with the following properties*:

(i) $\tilde{I}_k\subseteqq[0, 1]$

(ii) $\text{mes}\,(\cup_{k=1}^{M}\tilde{I}_k\cap E)<\varepsilon/3$

(iii) mes $(\cap_{k=1}^{M} \tilde{I}_k^c \cap I_0 \cap E^c) < \varepsilon/3$.

Then, $\int_0^1 I_E(x) \exp(2n\pi ix)dx = C_n$ *tends to zero as n tends to* ∞.

Proof. Considering the conditions (ii) (iii), let's choose N such that $2^{M+1}/N < \varepsilon/3$. Then

$$\left| \int_0^1 I_E(x) \exp(2n\pi ix)dx \right| = \left| \int_0^{1/n} \tilde{I}_E(x) \exp(2n\pi ix)dx \right|$$
$$< (lA/2n + 2\varepsilon/3) - (lA/2n - \varepsilon/3) = \varepsilon$$

is valid for $n \geqq N$, where $\tilde{I}_E(x) = \sum_{k=0}^{n-1} I_E(x-k/n) \cdot I_{[0,1/n]}(x)$. Here, from the uniformly l th covering of the interval $[0, 1/n]$ effected by the periodicity, $lA/2n$ is derived $\left(A = \int_0^{1/2} \sqrt{2} \sin 2\pi x dx\right)$; from the positive effect related to the end of $\tilde{I}_k (k \leqq M)$ and the condition (ii), $2\varepsilon/3$ is derived; and from the negative effect related to condition (iii), $\varepsilon/3$ is derived. Hence, Riemann-Lebesgue's theorem is satisfied for this $I_E(x)$.

Corollary 1. *The closed set contained in* $[0, 1]$ *satisfies the conditions of Theorem* 1.

Lemma 2. *Suppose that irrational points* $x_1, x_2, \cdots, x_n$ *are contained in the interval* $[0, 1]$. *Then for any* $\varepsilon > 0$, *for any positive numbers* $a_1, a_2, \cdots, a_n$ *contained in* $[0, 1]$ *and for any positive integer* K *there exist positive integers* $k, n_1, n_2, \cdots, n_n$ *such that* $k > K$ *and* $|kx_i - n_i - a_i| < \varepsilon$ *for* $i = 1, \cdots, n$.

From this Lemma 2 etc. we assert the following

Corollary 2. *If* $f(x)$ *(even one contained in* L^1*) has* δ*-like singularities (by Dirac), then* $C_n = \int_0^1 f(x) \exp(2\pi nix)dx$ *must not tend to zero as n tends to* ∞.

From the above Corollary 2 it is easily seen that the essential points of natural cut-off are not to use infinite point set or nowhere dense perfect set but to use characteristic function of nowhere dense perfect set. For natural cut-off the Nakanishi's E. R. integral representation of δ function (to appear) is also effective, and it can be also used to represent the process tending to various physical models. But, since the order of C_n tending to zero must be $o(1/n)$ in our requirement, it seems that exact cut-off is needed and A-integral itself or E. R. integral itself becomes to very important one.

§ 3. The model representing the process tending to various physical models. At the first step, let's show the diagram of the processes tending to limit appearing in A-integral representation of δ-like functions.

$$\begin{array}{ccc} f_n(x) \cong d_n(x) = (1/2\pi)\int_{-n}^{n} e^{ikx}dk & \xrightarrow{n\to\infty} & f_\delta(x) \cong \delta(x) = (1/2\pi)\int_{-\infty}^{\infty} e^{ikx}dk \\ \uparrow m\to\infty & & \uparrow m\to\infty \\ f_{n,m}(x) = [f_n]_m(x) & & f_{\delta,m}(x) = [f_\delta]_m(x). \end{array}$$

Here, $f_n(x)$ and $f_\delta(x)$ are A-integrable functions in local, and $\cong$ means the equal in distribution's space, for example

$$(A)\int f_\delta(x)\varphi(x)dx=\int\delta(x)\varphi(x)dx \text{ for } \varphi(x)\in(D) \text{ (by L. Schwartz).}$$

A-integral representation has the following defects to the construction of our model. But, for the explanation of the complicated physical phenomena, it may also become to advantages. Namely

(1) $f_n(x)$ must take any large values,

(2) the domain of $f_n(x)$ (or $f_{nm}(x)$) must be contained in all real axis. (By using other regularization of δ function, this defect can be omitted.)

(3) the domain of $f_n(x)$ has the properties in § 1. Namely it is a dense set.

Furthermore we can also choose an approximate function (using for cut-off satisfying the condition $o(1/n)$) $f_{n,m(k)}(x)$ defined on nowhere dense perfect set which depends upon k in $\exp ikx$.

Lemma 3. *The domain of $f_n(x)$ must not be nowhere dense.*

If it is nowhere dense, then we cannot describe the small behavior of $d_n(x)$. Hence, the conclusion of this lemma is evident. In the following let's show the physical models used by us as the cases tending to limit. Using the source $f_n(x)$, the cut-off model is constructed. Using the source $f_{n,m}(x)$ (or $f_{\delta,m}(x)$) the causality model is constructed. Using the source $f_\delta(x)$ the fixed point source model is constructed. The above diagram related to A-integral representation shows the relation among them. It seems that the true form of elementary particles are the middle (or mixture) of these three models. A-integral representation shows this fact well.

E. R. representation of δ function (by Nakanishi) related to the characteristic function of nowhere dense perfect set $I_E(x)$ (or $I_E(x)$ itself) has the following advantages.

(1) It may take bounded values except for 0.

(2) It may contain in bounded intervals.

(3) Its domain is a nowhere dense perfect set.

But, it seems that this representation does not necessarily satisfy the condition such that the order of C_n is $o(1/n)$. If there is no need for satisfying this condition, this model and natural cut-off can be used.

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