11. On the Sequence of Fourier Coefficients

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1. Let $A:(d_{n,k})$, $n, k=0, 1, 2, \cdots$ and $d_{n,0}$, be a triangular Toeplitz matrix satisfying the conditions

(1.1) $$\lim_{n\to\infty} d_{n,k}=1 \quad \text{for every fixed } k,$$

and

(1.2) $$\sum_{k=0}^{n} |\Delta d_{n,k}| \leq K$$

where

$$\Delta d_{n,k}=d_{n,k}-d_{n,k+1}$$

and K being an absolute constant independent of n. It is easy to see that the third condition of Silverman Toeplitz theorem [page 64, 1] is automatically satisfied.

An infinite series $\sum u_n$ with the sequence of partial sum $\{S_n\}$ is said to be summable A to the sum S if

(1.3) $$\lim_{n\to\infty} \sum_{k=0}^{n} \Delta d_{n,k} S_k = S.$$

We obtain another method of summation viz. A. (C, 1) by superimposing the method A on the Cesàro mean of order one.

2. Let $f(x)$ be a function which is integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and is defined outside this by periodicity. Let the Fourier Series of $f(x)$ be

(2.1) $$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)=\frac{1}{2}a_0+\sum_{1}^{\infty}A_n(x),$$

then the conjugate series of (2.1) is

(2.2) $$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx)=\sum_{1}^{\infty}B_n(x).$$

We write

$$\psi(t)=f(x+t)+f(x-t)-l.$$

Siddiqui [4] has proved that, if

(2.3) $$\sum_{k=0}^{n} |\Delta^2 d_{n,k}| = o(1)$$

and $\psi(t)$ is of bounded variation in $(0, \pi)$, then $\{nB_n(x)\}$ is summable A to l at $t=x$. Recently he [5] gave a necessary and sufficient condition on A for the validity of the above theorem.

The object of this paper is to prove the following theorem:

Theorem. If

(2.4) $$\Psi(t)=\int_0^t |\psi(t)|\,dt=o\left(t/\log\frac{1}{t}\right) \quad \text{as } t\to 0,$$

and for some γ with $0<\gamma<1$,

(2.5) $$\lim_{n\to\infty}\sum_{k=0}^{n} K^{\gamma}\,|\Delta^2 d_{n,k}|=o(1),$$

then the sequence $\{nB_n(x)\}$ is summable $A\cdot(C,1)$ to the sum l/π.

It may be noted here that the regular matrics which satisfy the condition (2.5) with $\gamma=0$ are called strongly regular [2].

If we choose $\Delta d_{n,k}=\dfrac{1}{(n-k+1)\log n}$, summability A reduces to Harmonic summability. It is easy to see that (2.5) is also satisfied. Hence our theorem includes the result of Varsney [8] as a particular case.

3. Proof of the theorem. If we denote the $(C,1)$ transform of the sequence $\{nB_n(x)\}$ by ρ_n, we have after Mohanty and Nanda [3],

(3.1) $$\rho_n-l/\pi=\frac{1}{\pi}\int_0^{\delta}\psi(t)\left[\frac{\sin nt}{nt^2}-\frac{\cos nt}{t}\right]dt+o(1),$$

by Riemann-Lebesgue theorem, δ being constant greater than zero.

On account of regularity of the A method of summation. We need only to prove that

(3.2) $$I=\frac{1}{\pi}\sum_{\substack{k=1\\ \lim n\to\infty}}^{n}\Delta d_{n,k}\int_0^{\delta}\psi(t)g_n(t)dt=o(1),$$

where

(3.3) $$g_n(t)=\frac{\sin nt}{nt^2}-\frac{\cos nt}{t}.$$

We require the following inequalities which can be easily obtained by expanding sine and cosine in powers of n and t:

(3.4) $$g_n(t)=O(n^2t)$$
$$=O(n)$$

and

(3.5) $$g_n(t)=O(t^{-1}).$$

It is also known [7] that

(3.6) $$\sum_{\nu=1}^{n}\frac{\sin\nu}{\nu}=O(1).$$

Now, for $0<\gamma<1$,

$$I=\frac{1}{\pi}\sum_{k=1}^{n}\Delta d_{n,k}\left\{\int_0^{k^{-1}}+\int_{k^{-1}}^{k^{-\gamma}}+\int_{k^{-\gamma}}^{\delta}\right\}g_n(t)dt$$
$$=\frac{1}{\pi}\left\{\sum_{k=1}^{n}\Delta d_{n,k}(I_1+I_2+I_3)\right\}$$
$$=\frac{1}{\pi}[\tau_n(I_1)+\tau_n(I_2)+\tau_n(I_3)], \quad \text{say.}$$

Using (3.4), we get

$$\begin{aligned}|I_1| &\leq \int_0^{k^{-1}} |\psi(t)|\,|g_n(t)|\,dt \\ &\leq O(k)\Psi(1/k) \\ &= o(1/\log k)+o(1) \\ &= o(1) \quad \text{as } k\to\infty.\end{aligned}$$

Further, with the help of (3.5), we write

$$\begin{aligned}|I_2| &\leq O(1)\cdot\int_{k^{-1}}^{k^{-\gamma}} \frac{|\psi(t)|}{t}\,dt \\ &= O(1)\cdot\left\{\left[\frac{\Psi(t)}{t}\right]_{k^{-1}}^{k^{-\gamma}} + \int_{k^{-1}}^{k^{-\gamma}} \frac{\Psi(t)}{t^2}\,dt\right\} \\ &= o\left(\frac{1}{\log k}\right)+o(1)\cdot(\log\gamma) \\ &= o(1) \quad \text{as } k\to\infty.\end{aligned}$$

Thus the first two terms in (3.7) can be made as small as we please by choosing n sufficiently large as the transformation τ_n is regular.

From (3.3) and (3.5), we write [see also, 6]

$$\begin{aligned}G_\nu(t) &= g_1(t)+\cdots+g_\nu(t) \\ &= \frac{1}{t^2}\sum_{\nu=1}^{n}\frac{\sin\nu t}{\nu} - \frac{1}{t}\sum_{\nu=1}^{n}\cos\nu t \\ &= O\left(\frac{1}{t^2}\right) - \frac{1}{t}D_\nu(t) \\ &= O\left(\frac{1}{t^2}\right),\end{aligned}$$

where $D_\nu(t)$ is the Dirichlet Kernel for convergence of Fourier Series, and it is known that $D_\nu(t)=O\left(\frac{1}{t}\right)$. It is easy to see that $\sum|\Delta d_{n,k}|<\infty$ and $\sum k^\gamma|\Delta^2 d_{n,k}|<\infty$, imply that $k\Delta^\gamma d_{n,k}\to 0$, hence using Abel transformation, we write

$$\begin{aligned}|\tau_n(I_3)| &= \left|\sum_{k=1}^{n}\Delta d_{n,k}\int_{k^{-\gamma}}^{\delta}\psi(t)[G_k(t)-G_{k-1}(t)]dt\right| \\ &\leq \left|\sum_{k=1}^{n-1}\Delta^2 d_{n,k}\int_{k^{-\gamma}}^{\delta}\psi(t)G_k(t)dt\right| \\ &\quad + \left|\sum_{k=2}^{n}\Delta d_{n,k}\int_{k^{-\gamma}}^{(k-1)^{-\gamma}}\psi(t)G_{k-1}(t)dt\right| + o(1) \\ &= L_1+L_2, \quad \text{say.}\end{aligned}$$

$$\begin{aligned}L_1 &\leq O(1)\cdot\left\{\sum_1^{n-1}|\Delta^2 d_{n,k}|\int_{k^{-\gamma}}^{\delta}\frac{|\psi(t)|}{t^2}dt\right\} \\ &= O(1)\left\{\sum_1^{n-1}k^\gamma|\Delta^2 d_{n,k}|\int_{k^{-\gamma}}^{\delta}\frac{|\psi(t)|}{t}dt\right\} \\ &= O(1)\left\{\sum_1^{n-1}k^\gamma|\Delta^2 d_{n,k}|\right\}\cdot\left\{o\left(\frac{1}{\log k}\right)+o(\log\gamma)\right\}, \quad \text{by (3.5),} \\ &= o(1), \quad \text{with the hypothesis (2.5).}\end{aligned}$$

Further,

$$L_2=O(1)\cdot\left\{\sum_{k=2}^{n}|\Delta d_{n,k}|\int_{k^{-\gamma}}^{(k-1)^{-\gamma}}\frac{|\psi(t)|}{t^2}dt\right\}$$

$$=O(1)\left\{\sum_{k=2}^{n}k^{\gamma}|\Delta d_{n,k}|\int_{k^{-\gamma}}^{(k-1)^{-\gamma}}\frac{|\psi(t)|}{t}dt\right\}$$

$$=O(1)\left\{\sum_{k=2}^{n}k^{\gamma}|\Delta d_{n,k}|\right\}\left\{\left[\frac{\Psi(t)}{t}\right]_{k^{-\gamma}}^{(k-1)^{-\gamma}}-\int_{k^{-\gamma}}^{(k-1)^{-\gamma}}\frac{1}{t\log\frac{1}{t}}dt\right\}$$

$$=o(1).$$

This completes the proof of the theorem.

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