# 10. On the Absolute Nörlund Summability of a Factored Fourier Series

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1. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n=p_0+p_1+p_2+\cdots+p_n;\ P_{-1}=p_{-1}=0.$$

The sequence to sequence transformation

$$t_n=\frac{1}{P_n}\sum_{\nu=0}^{n}p_{n-\nu}s_\nu=\frac{1}{P_n}\sum_{\nu=0}^{n}p_\nu s_{n-\nu}\quad(P_n\neq 0)\tag{1.1}$$

defines the sequence $\{t_n\}$ of Nörlund mean [3] of the sequence $\{s_n\}$, generated by the sequence of coefficient $\{p_n\}$. The series $\sum a_n$ is said to be absolutely Nörlund summable, or *summable* $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation, that is, the series $\sum|t_n-t_{n-1}|$ is convergent [2]. In the special case in which

$$p_n=\frac{1}{n+1},\tag{1.2}$$

and therefore

$$P_n\sim\log n,\quad \text{as } n\to\infty,$$

the Nörlund mean reduces to the familiar harmonic mean [5]. Thus summability $|N, p_n|$, where $p_n$ is defined as in (1.2), is the same as summability $\left|N, \frac{1}{n+1}\right|$.

Let $f(t)$ be a periodic function with period $2\pi$ and integrable $(L)$ over $(-\pi, \pi)$. Without any loss of generality the constant term in the Fourier series can be taken to be zero, that is,

$$\int_{-\pi}^{\pi}f(t)dt=0$$

and

$$f(t)\sim\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$$
$$\equiv\sum_{n=1}^{\infty}A_n(t).$$

We write

$$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\},$$

$$\phi_1(t)=\frac{1}{t}\int_0^t\phi(u)du.$$

2. Varshney [7] has considered the following theorem to study the absolute harmonic summability of the factored Fourier series. He proved:

THEOREM A. If $\phi(t)$ is of bounded variation in $(0, \pi)$, then the series $\sum A_n(t)/\log(n+1)$ is absolutely harmonic summable.

Generalising the above theorem of Vershney Singh [6] has recently established the following theorem under Jordan's criterion of convergence of the Fourier series.

THEOREM B. If $\phi(t)$ is of bounded variation in $(0, \pi)$, then the series

$$\sum \frac{(n+1)p_n}{p_n} A_n(t)$$

is summable $|N, p_n|$, at $t=x$, where the sequence $\{p_n\}$ is real non-negative and non-increasing such that

(i) $\left\{\frac{(n+1)p_n}{p_n}\right\}$ is a sequence of bounded variation and

(ii) the sequence $\{p_n - p_{n+1}\}$ is non-increasing.

The object of the present paper is to study the Nörlund summability of the factored Fourier series under de la Vallée Poussin criterion of convergence of the Fourier series, which is known to be less stringent than Jordan's criterion of convergence. In what follows we establish the following:

THEOREM 1. *If $\phi_1(t)$ is of bounded variation in $(0, \pi)$, then the series*

$$\sum \frac{P_n \lambda_n}{n} A_n(t),$$

*where $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, is absolutely Nörlund summable, or summable $|N, p_n|$, where $\{p_n\}$ is non-negative, non-increasing such that $\{p_n - p_{n+1}\}$ is non-increasing.*

It may be remarked that the above Theorem 1 generalises the following theorems of Lal [1] and Prasad and Bhatt [4] respectively.

THEOREM C. If $\phi_1(t)$ is of bounded variation in $(0, \pi)$, then the series

$$\sum \frac{\log(n+1)\lambda_n}{n} A_n(t),$$

where $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, is absolutely harmonic summable.

THEOREM D. If $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, and $\phi_1(t)$ is of bounded variation in $(0, \pi)$, then the series $\sum \lambda_n A_n(t)$, at $t=x$, is summable $|C, 1|$.

In order to prove Theorem 1 we establish the following theorem for factored infinite series:

THEOREM 2. *If*

$$B_n=\sum_{\nu=1}^{n}\nu a_\nu=O(n),$$

*then the series* $\sum P_n\lambda_n a_n/n$, *where* $\{\lambda_n\}$ *is a convex sequence such that* $\sum n^{-1}\lambda_n$ *is convergent, is absolutely Nörlund summable, or summable* $|N, p_n|$, *where* $\{p_n\}$ *is non-negative, non-increasing such that* $\{p_n-p_{n+1}\}$ *is non-increasing.*

3. We require the following lemma for the proof of our Theorem:

Lemma 1. *If* $\phi_1(t)$ *is of bounded variation in* $(0,\pi)$, then

$$\frac{1}{n}\sum_{\nu=1}^{n}\nu A_\nu(x)=O(1),$$

*as* $n\to\infty$.

This lemma is a particular case of a lemma due to Prasad and Bhatt [4].

PROOF OF THEOREM 2. Since from (1.1)

$$t_n=\sum_{\nu=0}^{n}\frac{P_\nu u_{n-\nu}}{P_n},$$

where

$$u_n=\frac{P_n\lambda_n a_n}{n},$$

we have

$$\begin{aligned}t_n-t_{n-1}&=\sum_{\nu=0}^{n-1}\left(\frac{P_\nu}{P_n}-\frac{P_{\nu-1}}{P_{n-1}}\right)u_{n-\nu}\\&=\frac{1}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}(P_np_\nu-P_\nu p_n)u_{n-\nu}\\&=\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-1}(P_np_{n-k-1}-P_{n-k-1}p_n)u_{k+1}\\&=\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-1}P_n(p_{n-k-1}-p_n)\frac{P_{k+1}\lambda_{k+1}a_{k+1}}{k+1}\\&\quad+\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-1}p_n(P_n-P_{n-k-1})\frac{P_{k+1}\lambda_{k+1}a_{k+1}}{k+1}\\&=\sum\nolimits_1+\sum\nolimits_2,\end{aligned}$$

say.

By Abel's transformation

$$\begin{aligned}\sum\nolimits_1&=\frac{1}{P_{n-1}}\sum_{k=0}^{n-1}(p_{n-k-1}-p_n)\frac{P_{k+1}\lambda_{k+1}}{(k+1)^2}(k+1)a_{k+1}\\&=\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}\Delta\left\{(p_{n-k-1}-p_n)\frac{P_{k+1}\lambda_{k+1}}{(k+1)^2}\right\}\sum_{\mu=0}^{k}(\mu+1)a_{\mu+1}\\&\quad+\frac{1}{P_{n-1}}\left\{(p_0-p_n)\frac{P_n\lambda_n}{n^2}\right\}\sum_{\mu=0}^{n-1}(\mu+1)a_{\mu+1}\end{aligned}$$

$$
\begin{aligned}
&=O\left[\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}|p_{n-k-1}-p_{n-k-2}|\frac{P_{k+1}\lambda_{k+1}}{(k+1)^2}|B_{k+1}|\right]\\
&\quad+O\left[\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-1}-p_n)\frac{p_{k+1}\lambda_{k+1}}{(k+1)^2}|B_{k+1}|\right]\\
&\quad+O\left[\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-1}-p_n)\frac{P_{k+1}\Delta\lambda_{k+1}}{(k+1)^2}|B_{k+1}|\right]\\
&\quad+O\left[\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-1}-p_n)\frac{P_{k+1}\lambda_{k+1}}{(k+1)^3}|B_{k+1}|\right]+O\left[\frac{\lambda_n}{n}\right]\\
&=O\left[\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}|p_{n-k-1}-p_{n-k-2}|\frac{P_{k+1}\lambda_{k+1}}{k+1}\right]\\
&\quad+O\left[\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-1}-p_n)\frac{P_{k+1}\lambda_{k+1}}{k+1}\right]\\
&\quad+O\left[\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-1}-p_n)\frac{P_{k+1}\Delta\lambda_{k+1}}{k+1}\right]\\
&\quad+O\left[\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-1}-p_n)\frac{P_{k+1}\lambda_{k+1}}{(k+1)^2}\right]+O\left[\frac{\lambda_n}{n}\right]\\
&=O\left[\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}|p_{n-k-1}-p_{n-k-2}|\frac{P_{k+1}\lambda_{k+1}}{k+1}\right]\\
&\quad+O\left[\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-1}-p_n)\frac{P_{k+1}\Delta\lambda_{k+1}}{k+1}\right]+O\left[\frac{\lambda_n}{n}\right]\\
&=O[M_1]+O[M_2]+O\left[\frac{\lambda_n}{n}\right],
\end{aligned}
$$

say, since

$$
\begin{aligned}
p_{n-k-1}-p_n&\leq(k+1)(p_{n-k-1}-p_{n-k})\\
&\leq(k+1)(p_{n-k-2}-p_{n-k-1})
\end{aligned}
$$

and

$$(k+1)p_{k+1}\leq P_{k+1}.$$

Now

$$
\begin{aligned}
\sum_{n=2}^{\infty}|M_1|&=\sum_{n=2}^{\infty}\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-2}-p_{n-k-1})\frac{P_{k+1}\lambda_{k+1}}{k+1}\\
&=\sum_{k=0}^{\infty}P_{k+1}\frac{\lambda_{k+1}}{k+1}\sum_{n=k+2}^{\infty}(p_{n-k-2}-p_{n-k-1})\frac{1}{P_{n-1}}\\
&\leq O(1)\sum_{k=0}^{\infty}P_{k+1}\frac{\lambda_{k+1}}{k+1}\frac{1}{P_{k+1}}\\
&=O(1)\sum_{k=0}^{\infty}\frac{\lambda_{k+1}}{k+1}=O(1).
\end{aligned}
$$

*Again*

$$\begin{aligned}\sum_{n=2}^{\infty}|M_2| &= \sum_{n=2}^{\infty}\frac{1}{P_{n-1}}\sum_{k=2}^{n-2}(p_{n-k-1}-p_n)\frac{P_{k+1}\Delta\lambda_{k+1}}{k+1}\\ &\leq \sum_{n=2}^{\infty}\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-1}-p_{n-k})P_{k+1}\Delta\lambda_{k+1}\\ &= \sum_{k=0}^{\infty}P_{k+1}\Delta\lambda_{k+1}\sum_{n=k+2}^{\infty}(p_{n-k-1}-p_{n-k})\frac{1}{P_{n-1}}\\ &\leq O(1)\sum_{k=0}^{\infty}P_{k+1}\Delta\lambda_{k+1}\frac{1}{P_{k+1}}=O(1)\sum_{k=0}^{\infty}\Delta\lambda_{k+1}=O(1),\end{aligned}$$

and by hypothesis

$$\sum\frac{\lambda_n}{n}=O(1).$$

Now, by Abel's transformation

$$\begin{aligned}\sum\nolimits_2 &= \frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-1}(P_n-P_{n-k-1})\frac{P_{k+1}\lambda_{k+1}}{(k+1)^2}(k+1)a_{k+1}\\ &= \frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}\Delta\left\{(P_n-P_{n-k-1})\frac{P_{k+1}\lambda_{k+1}}{(k+1)^2}\right\}\sum_{\mu=0}^{k}(\mu+1)a_{\mu+1}\\ &\quad+\frac{p_n}{P_nP_{n-1}}\left\{(P_n-P_0)\frac{P_n\lambda_n}{n^2}\right\}\sum_{\mu=0}^{n-1}(\mu+1)a_{\mu+1}\\ &= O\left[\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}(P_{n-k-1}-P_{n-k-2}\frac{P_{k+1}\lambda_{k+1}}{(k+1)^2}|B_{k+1}|\right]\\ &\quad+O\left[\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}(P_n-P_{n-k-1})\frac{P_{k+1}\lambda_{k+1}}{(k+1)^2}|B_{k+1}|\right]\\ &\quad+O\left[\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}(P_n-P_{n-k-1})\frac{P_{k+1}\Delta\lambda_{k+1}}{(k+1)^2}|B_{k+1}|\right]\\ &\quad+O\left[\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}(P_n-P_{n-k-1})\frac{P_{k+1}\lambda_{k+1}}{(k+1)^3}|B_{k+1}|\right]+O\left[\frac{\lambda_n}{n}\right]\\ &= O\left[\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}p_{n-k-1}\frac{P_{k+1}\lambda_{k+1}}{k+1}\right]\\ &\quad+O\left[\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}(P_n-P_{n-k-1})\frac{p_{k+1}\lambda_{k+1}}{k+1}\right]\\ &\quad+O\left[\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}(P_n-P_{n-k-1})\frac{P_{k+1}\Delta\lambda_{k+1}}{k+1}\right]\\ &\quad+O\left[\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}(P_n-P_{n-k-1})\frac{P_{k+1}\lambda_{k+1}}{(k+1)^2}\right]+O\left[\frac{\lambda_n}{n}\right]\\ &= O\left[\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}p_{n-k-1}\frac{P_{k+1}\lambda_{k+1}}{k+1}\right]\\ &\quad+O\left[\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}(P_n-P_{n-k-1})\frac{P_{k+1}\Delta\lambda_{k+1}}{k+1}\right]+O\left[\frac{\lambda_n}{n}\right]\\ &= O[N_1]+O[N_2]+O\left[\frac{\lambda_n}{n}\right],\end{aligned}$$

say, since

$$P_n - P_{n-k-1} \le (k+1)p_{n-k} \le (k+1)p_{n-k-1}$$

and

$$(k+1)p_{k+1} \le P_{k+1}.$$

We have

$$\sum_{n=2}^{\infty} |N_1| = \sum_{n=2}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=0}^{n-2} p_{n-k-1} \frac{P_{k+1}\lambda_{k+1}}{k+1}$$

$$= \sum_{k=0}^{\infty} P_{k+1} \frac{\lambda_{k+1}}{k+1} \sum_{n=k+2}^{\infty} p_{n-k-1} \frac{p_n}{P_n P_{n-1}}$$

$$\le O(1) \sum_{k=0}^{\infty} P_{k+1} \frac{\lambda_{k+1}}{k+1} \frac{1}{P_{k+1}}$$

$$= O(1),$$

$$\sum_{n=0}^{\infty} |N_2| = \sum_{n=2}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=0}^{n-2} (P_n - P_{n-k-1}) \frac{P_{k+1}\Delta\lambda_{k+1}}{k+1}$$

$$\le \sum_{n=2}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=0}^{n-2} p_{n-k} P_{k+1} \Delta\lambda_{k+1}$$

$$= \sum_{k=0}^{\infty} P_{k+1}\Delta\lambda_{k+1} \sum_{n=k+2}^{\infty} p_{n-k} \frac{p_n}{P_n P_{n-1}}$$

$$\le O(1) \sum_{k=0}^{\infty} P_{k+1}\Delta\lambda_{k+1} \frac{1}{P_{k+1}}$$

$$= O(1) \sum_{k=0}^{\infty} \Delta\lambda_{k+1} = O(1),$$

and by hypothesis

$$\sum \frac{\lambda_n}{n} = O(1).$$

Hence we establish that

$$\sum |t_n - t_{n-1}| < \infty,$$

which proves the Theorem.

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