2. A Note on General Connections

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Let M be an n-dimensional differentiable manifold and γ be a general connection on M. In terms of local coordinates u^i of M, γ can be written as:

(1) $$\gamma=\partial u_j\otimes(P_i^j d^2u^i+\Gamma_{ih}^j du^i\otimes du^h),$$ [1]

where $\partial u_j=\partial/\partial u^j$ and d^2u^i is the differential of order 2 of u^i. As easily shown, P_i^j are the components of a tensor field of type (1, 1), which we denote by

(2) $$P=\partial u_j\otimes P_i^j du^i=\lambda(\gamma).$$

For any tensor field Q of type (1, 1) with local components Q_i^j and a general connection γ, we can define two general connections as

(3) $$Q\gamma=\partial u_k Q_j^k\otimes(P_i^j d^2u^i+\Gamma_{ih}^j du^i\otimes du^h)$$ [2]

and

(4) $$\gamma Q=\partial u_j\otimes(P_k^j d(Q_i^k du^i)+\Gamma_{kh}^j(Q_i^k du^i)\otimes du^h).$$

On these multiplications of general connections and tensor fields of type (1, 1), we have

Lemma 1. *The products of general connections and tensor fields of type* (1, 1) *defined by* (3) *and* (4) *satisfy the associative law with respect to the multiplications from the left and the right.*[3]

When $\lambda(\gamma)=1$, γ is an affine connection on M and when $\lambda(\gamma)=0$, γ is a tensor field of type (1, 2). In this note, we investigate the condition that for a given general connection γ on M there exist affine connections γ_1 such that $\gamma=P\gamma_1$ or $\gamma=\gamma_1 P$, where $P=\lambda(\gamma)$. The following theorem is evident.

Theorem A. *Let γ be a general connection on M and put $P=\lambda(\gamma)$. In order to exist an affine connection γ_1 such that*

(5) $$\gamma=P\gamma_1 \quad (or\ \gamma=\gamma_1 P)$$

it is necessary and sufficient that for an affine connection γ^* on M, the tensor $T=\gamma-P\gamma^*$ (or $\gamma-\gamma^*P$) is of the form $T_{ih}^j=P_l^j V_{ih}^l$ (or $V_{lh}^j P_i^l$), where T_{ih}^j and V_{ih}^j are the local components of T and a tensor of type (1, 2).

Lemma 2. *Let V be an n-dimensional vector space over the real field R and P: $V\to V$ be a linear transformation whose minimal polynomial is*

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1) See [3]
2) See [7]
3) See Proposition 1.1 in [7]

(6) $$F(\xi)=\xi^l+a_1\xi^{l-1}+\cdots+a_{l-r}\xi^r,\ a_{l-r}\neq 0.$$

Then we have:

(i) *When and only when $r=0, P$ is regular, i.e. an isomorphism.*

(ii) *When $r>0, P^t$ is normal if and only if $t\geqq r$, that is*

$$V=\text{image of } P^t\oplus\text{kernel of } P^t.$$

(iii) *Furthermore, if $l>r$, putting*

(7) $$g(\xi)=-\frac{1}{a_{l-r}}\sum_{\alpha=0}^{l-r-1}a_\alpha\xi^{l-r-1-\alpha}\quad(a_0=1),$$

P and $g(P)$ are the inverse of each other on $P^r(V)$.

Proof. (i) is evident from the fact the eignvalues of P is roots of the equation $f(\xi)=0$. In the case, we have

$$P\cdot g(P)=1.$$

On (iii), the identity:

$$f(P)=P^l+a_1P^{l-1}+\cdots+a_{l-r}P^r=0$$

can be written as

(8) $$P\cdot g(P)\cdot P^r=P^r.$$

Since we have

$$P(P^r(V))\subset P^r(V)\quad\text{and}\quad g(P)(P^r(V))\subset P^r(V),$$

(8) shows that P and $g(P)$ are the inverse of each other on $P^r(V)$. Accordingly we have

(9) $$P^r(V)=P^{r+1}(V)=\cdots$$

and $P^{r-1}(V)\supsetneqq P^r(V)$, for otherwise $P\mid P^{r-1}(V)$ is an isomorphism on $P^{r-1}(V)$, and so its minimal polynomial has a non-zero constant term by (i) and the polynomial $\times\xi^{r-1}$ is divisible by $f(\xi)$. Hence we have

(10) $$V\supsetneqq P(V)\supsetneqq\cdots\supsetneqq P^{r-1}(V)\supsetneqq P^r(V)=P^{r+1}(V)=\cdots$$

Regarding (ii), in general, a linear transformation $Q: V\rightarrow V$ is normal if and only if Q is isomorphic on $Q(V)$. In the both cases $l>r$ and $l=r$, (10) holds good. Hence for an integer t, P^t is normal if and only if $t\geqq r$. q.e.d.

Theorem B. *Let γ be a general connection on M such that: the minimal polynomial of $P=\lambda(\gamma)$ is given by*

(11) $$f(\xi)=\xi^l+a_1\xi^{l-1}+\cdots+a_{l-r}\xi^r,$$

where $a_1,\cdots,a_{l-r}$ are differentiable and everywhere $a_{l-r}\neq 0$, and the dimension of the image of P is constant.

In order to exist an affine connection γ_1 such that

(5) $$\gamma=P\gamma_1\quad(\textit{or }\gamma=\gamma_1P),$$

it is necessary that (i) $r=0$ *or* (ii) $r>0$ *and*

(12) $$(P^{l-1}+a_1P^{l-2}+\cdots+a_{l-r}P^{r-1})\gamma=0$$

(*or*

(13) $$\gamma(P^{l-1}+a_1P^{l-2}+\cdots+a_{l-r}P^{l-1})=0).$$

When $r=1$, (12) (*or* (13)) *is sufficient.*

Proof. When $r=0$, P is regular and the connection $\gamma_1=P^{-1}\gamma$ is the solution of (5) by Lemma 1.

When $r>0$, clearly (12) is a necessary condition that (5) has solutions.

In the case $r=1$, if $l=r$, then $P=0$ and so (5) become trivial. Let assume $l>r=1$. Putting

$$g(\xi)=-\frac{1}{a_{l-1}}\sum_{\alpha=0}^{l-2} a_\alpha \xi^{l-2-\alpha} \quad (a_0=1), \tag{14}$$

then the identity $f(P)=0$ can be written as

$$P\cdot g(P)\cdot P=P. \tag{15}$$

By Lemma 2, P is normal. Let M_x be the tangent space to M at $x\in M$, P_x and N_x be the image and the kernel of P on M_x respectively. Then we have the direct sum:

$$M_x=P_x\oplus N_x,$$

by which we define two normal tensor fields A and N such that A: $M_x\rightarrow P_x$ and N: $M_x\rightarrow N_x$ are projections. We have easily

$$A+N=1,\ AN=NA=0,\ A^2=A \text{ and } N^2=N.$$

Now, we define a general connection $\bar{\gamma}$ by

$$\bar{\gamma}=g(P)\gamma,$$

then we have by (12) the relation:

$$P\bar{\gamma}=Pg(P)\gamma=\gamma,$$

$$\lambda(\bar{\gamma})=g(P)P \quad\text{and}\quad g(P)P\mid P_x=1\mid P_x.$$

Let Q be the tensor field defined by

$$Q\mid P_x=g(P)\mid P_x,\ Q(N_x)=0,$$

then we have clearly

$$QP=PQ=A,\ AP=PA=P,\ NP=PN=0$$

and

$$\lambda(A\bar{\gamma})=Ag(P)P=Q(Pg(P)P)=QP=A.$$

Now, taking a suitable affine connection $\bar{\gamma}_1$ on M, we define a general connection γ^* by

$$\gamma^*=A\bar{\gamma}+N\bar{\gamma}_1.$$

Then, we have

$$\lambda(\gamma^*)=\lambda(A\bar{\gamma})+\lambda(N\bar{\gamma}_1)=A+N=1,$$

hence γ^* is an affine connection. Furthermore

$$P\gamma^*=PA\bar{\gamma}+PN\bar{\gamma}_1=PA\bar{\gamma}=P\bar{\gamma}=\gamma.$$

γ^* is a solution of (5). Analogously, we can prove this for the equation $\gamma=\gamma_1 P$. q.e.d.

Theorem C. *Let γ be a general connection on M such that: The minimal polynomial of $P=\lambda(\gamma)$ is given by*

$$f(\xi)=\xi^l+a_1\xi^{l-1}+\cdots+a_{l-r}\xi^r, \tag{11}$$

where $l>r>1$, $a_1, \cdots, a_{l-r}$ are differentiable and everywhere $a_{l-r}\neq 0$ and the dimension of the image of P^r is constant. Let A and N be the tensor fields of type (1, 1) *defined by means of the direct sum* *decomposition*: $M_x=(P^r)_x\oplus N_x$, $x\in M$, *where* $(P^r)_x$ *and* N_x *are the image and the kernel of* P^r *on* M_x *respectively. Assuming that* γ *satisfies* (12) *and putting*

$$g(\xi)=-\frac{1}{a_{l-r}}\sum_{\alpha=0}^{l-r-1}a_\alpha\xi^{l-r-1-\alpha}\quad(a_0=1), \tag{16}$$

define a general connection $\tilde{\gamma}$ *by*

$$\tilde{\gamma}=\gamma-P(Ag(P)\gamma),$$

then the following hold:

(i) $P^{r-1}\tilde{\gamma}=0$.

(ii) $\lambda(Ag(P)\gamma)=A$ *and* $\lambda(\tilde{\gamma})=\tilde{P}=PN$.

(iii) *The minimal polynomial of* $\tilde{P}$ *is* ξ^r.

(iv) *The existence of an affine connection* γ_1 *such that* $\gamma=P\gamma_1$ *is equivalent to the one of an affine connection* $\tilde{\gamma}_1$ *such that* $\tilde{\gamma}=\tilde{P}\tilde{\gamma}_1$.

Proof. The identity

$$f(P)=P^l+a_1P^{l-1}+\cdots+a_{l-r}P^r=0$$

can be written as

$$P\cdot g(P)\cdot P^r=P^r.$$

From (12), we get

$$P^{r-1}(\gamma-P(g(P)\gamma))=0. \tag{17}$$

Since we have

$$\lambda(g(P)\gamma)=g(P)\lambda(\gamma)=g(P)P,$$

for any $x\in M$ we have

$$\lambda(g(P)\gamma)\mid(P^r)_x=A\mid(P^r)_x.$$

Now we define a tensor field Q of type (1, 1) on M by

$$Q\mid(P^r)_x=(g(P))^r\mid(P^r)_x \text{ and } Q(N_x)=0,$$

then we have

$$QP^r=P^rQ=A,\ AP^r=P^rA=P^r;\ NP^r=P^rN=0.$$

Hence, by virtue of Lemma 1, (17) can be written as

$$P^{r-1}(\gamma-P(Ag(P)\gamma))=0, \text{ that is } P^{r-1}\tilde{\gamma}=0.$$

Regarding (ii), we have easily

$$\lambda(A(g(P)\gamma))=Ag(P)P=Q(P^rg(P)P)=QP^r=A,$$
$$\lambda(\tilde{\gamma})=\lambda(\gamma-P(Ag(P)\gamma))=P-PA=PN=\tilde{P}.$$

Nextly, since N_x is the kernel of P^r on M_x, $x\in M$, we have

$$P(N_x)\subset N_x, \text{ that is } PN(M_x)\subset N(M_x),$$

from which we get easily

$$(PN)^k(M_x)=P^kN(M_x),\ k=2,3,\cdots$$

and especially

$$(PN)^r(M_x)=0.$$

If we have $(PN)^k=0$ for $k<r$, then from the above we get

$$N_x\subset \text{kernel of } P^k \text{ on } M_x\subset \text{kernel of } P^r \text{ on } M_x,$$

accordingly

$$\text{kernel of } P^k \text{ on } M_x=\text{kernel of } P^r \text{ on } M_x,$$

from which

$$\text{image of } P^k = \text{image of } P^r.$$

This contradicts to Lemma 2. Hence, the minimal polynomial of $\tilde{P}$ is ξ^r.

Lastly, assuming that there exists an affine connection γ_1 such that $\gamma = P\gamma_1$, we have

$$\tilde{\gamma} = \gamma - P(Ag(P)\gamma) = (P - PAg(P)P)\gamma_1 = P(1 - Ag(P)P)\gamma_1 = \\ = P(1-A)\gamma_1 = PN\gamma_1 = \tilde{P}\gamma_1.$$

Conversely, assuming that there exists an affine connection $\tilde{\gamma}_1$ such that $\tilde{\gamma} = PN\tilde{\gamma}_1$, we have

$$\gamma = \tilde{\gamma} + P(Ag(P)\gamma) = P(Ag(P)\gamma + N\tilde{\gamma}_1)$$

and

$$\lambda(Ag(P)\gamma + N\tilde{\gamma}_1) = A + N = 1.$$

Hence $\gamma_1 = Ag(P)\gamma + N\tilde{\gamma}_1$ is an affine connection. q.e.d.

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