35. A Note on Countable-dimensional Metric Spaces

By Keiô NAGAMI and J. H. ROBERTS

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This paper is a supplementary note to the characterization of countable-dimensional metric spaces by J. Nagata [2]. A space is *countable-dimensional* if it is the countable sum of zero-dimensional (in the sense of the covering dimension) subsets. A space is *strongly countable-dimensional* if it is the countable sum of finite dimensional closed subsets. Now Nagata has characterized these two classes of infinite dimensional metric spaces as follows:

Theorem A [2, Theorem 2.3]. *A metric space is countable-dimensional if and only if for every collection* $\{U_\alpha: \alpha<\tau\}$ *of open sets and every collection* $\{F_\alpha: \alpha<\tau\}$ *of closed sets such that* $F_\alpha\subset U_\alpha$, $\alpha<\tau$, *and such that* $\{U_\beta: \beta<\alpha\}$ *is locally finite for every* $\alpha<\tau$, *there exists a collection of open sets* V_α, $\alpha<\tau$, *satisfying*

i) $F_\alpha\subset V_\alpha\subset U_\alpha$, $\alpha<\tau$,

ii) order $(x, \mathrm{B}(\mathfrak{V}))<\infty$ *for every* $x\in X$, *where* $\mathfrak{V}=\{V_\alpha: \alpha<\tau\}$ *and* $\mathrm{B}(\mathfrak{V})=\{\mathrm{B}(V_\alpha)=\bar{V}_\alpha-V_\alpha: \alpha<\tau\}$.

Theorem B [2, Theorem 5.3]. *A metric space* X *is strongly countable-dimensional if and only if there exists a sequence* $\mathfrak{U}_1>\mathfrak{U}_2^*>\mathfrak{U}_2>\mathfrak{U}_3^*>\cdots$ *of open coverings* $\mathfrak{U}_i$ *of* X *such that*

i) *for* $x\in X$, $\{\mathrm{St}(x, \mathfrak{U}_i): i=1,2,\cdots\}$ *is a local base of* x,

ii) *for* $x\in X$, sup order $(x, \mathfrak{U}_i)<\infty$.

Our supplementary theorems to these are as follows:

Theorem 1. *A metric space* X *is countable-dimensional if and only if for every sequence of pairs of disjoint closed sets* $C_1, C_1'; C_2, C_2';\cdots$, *there exist separating closed sets* B_i *between* C_i *and* C_i', $i=1,2,\cdots$, *such that* $\{B_i: i=1,2,\cdots\}$ *is point-finite.*

The *only if* part of this theorem is a special case of Nagata [2, Lemma 2.1].

Theorem 2. *A metric space* X *is strongly countable-dimensional if and only if there exists a sequence* $\mathfrak{U}_1>\mathfrak{U}_2>\cdots$ *of open coverings* $\mathfrak{U}_j$ *of* X *such that*

i) *for* $x\in X$, $\{\mathrm{St}(x, U_i^4): i=1,2,\cdots\}$ *is a local base of* x,

ii) *for* $x\in X$, sup order $(x, \mathfrak{U}_i)<\infty$.

To prove Theorem 2 we need the following theorem for finite dimensional spaces.

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Theorem 3. *A metric space X has* $\dim X \leqq n$ *if there exists a sequence* $\mathfrak{U}_1 > \mathfrak{U}_2 > \cdots$ *of open coverings* $\mathfrak{U}_i$ *of* X *such that*

i) *for* $x \in X$, $\{\mathrm{St}(x, \mathfrak{U}_i^4): i=1,2,\cdots\}$ *is a local base of* x,

ii) order $\mathfrak{U}_i \leqq n+1$.

This is a generalization of Petr Vopĕnka's theorem [3]: A metric space X has $\dim X \leqq n$ if there exists a sequence $\mathfrak{U}_1 > \mathfrak{U}_2 > \cdots$ of open coverings $\mathfrak{U}_i$ of X such that i) lim mesh $\mathfrak{U}_i = 0$, ii) for every i, order $\mathfrak{U}_i \leqq n+1$.

Let K_ω be the subset of Hilbert cube which consists of all points $x=(x_1, x_2, \cdots)$ such that $x_i \neq 0$ for at most a finite number of values of i. Then K_ω is evidently strongly countable-dimensional. Nagata [2, Corollary 5.5] showed that K_ω is universal for the class of all strongly countable-dimensional, separable metric spaces. Now K_ω has the following property.

Theorem 4. *K_ω has no metric completion which is even countable-dimensional.*

It has been stated that E. Sklyarenko proved the non-existence of a countable-dimensional metric compactification of K_ω.

Our final result is as follows.

Theorem 5. *Let X be a countable-dimensional, compact metric space with* $\dim X = \infty$. *Then for any non-negative integer n there exists a closed subset F_n of X with* $\dim F_n = n$.

To prove Theorem 1 we need the following characterization theorem which is a very slight modification of a theorem due to Nagata [2, Theorem 2.2].

Theorem C. *A metric space X is countable-dimensional if and only if there exists a σ-locally finite open base $\mathfrak{V}$ such that* $\mathrm{B}(\mathfrak{V})$ *is point-finite.*

Proof of Theorem 1. Suppose the condition is satisfied. For any positive integer i there exists an open covering $\mathfrak{U}_i = \bigcup_{j=1}^{\infty} \mathfrak{U}_{ij}$, $\mathfrak{U}_{ij} = \{U_\alpha : \alpha \in A_{ij}\}$, of X and a closed covering $\mathfrak{F}_i = \bigcup_{j=1}^{\infty} \mathfrak{F}_{ij}$, $\mathfrak{F}_{ij} = \{F_\alpha : \alpha \in A_{ij}\}$, such that

i) mesh $\mathfrak{U}_i < 1/i$,

ii) $F_\alpha \subset U_\alpha$ for every $\alpha \in \bigcup_{j=1}^{\infty} A_{ij}$,

iii) every $\mathfrak{U}_{ij}$ is discrete.

Write $U_{ij} = \cup \{U_\alpha : \alpha \in A_{ij}\}$ and $F_{ij} = \cup \{F_\alpha : \alpha \in A_{ij}\}$. Then there exist open sets V_{ij}, $i,j=1,2,\cdots$, such that

i) $F_{ij} \subset V_{ij} \subset \bar{V}_{ij} \subset U_{ij}$ for every i and j,

ii) $\{\mathrm{B}(V_{ij}): i,j=1,2,\cdots\}$ is point-finite.

For every $\alpha \in A_{ij}$, set $V_\alpha = V_{ij} \cap U_\alpha$. Then $\mathfrak{V} = \{V_\alpha : \alpha \in \bigcup_{i,j} A_{ij}\}$ is a σ-discrete open base of X such that $\mathrm{B}(\mathfrak{V})$ is point-finite. By Theorem

C, X is countable-dimensional.

Proof of Theorem 3. Let $\mathfrak{U}_i=\{U_\alpha: \alpha\in A_i\}$, $i=1,2,\cdots$, be open coverings of X which satisfy the condition of the theorem. Let $f_i^{i+1}: A_{i+1}\to A_i$ be a function such that $f_i^{i+1}(\alpha)=\beta$ yields $U_\alpha\subset U_\beta$. For each pair $i>j$ let $f_j^i=f_j^{j+1}\cdots f_{i-1}^i$ and f_i^i the identity mapping. Let $\mathfrak{G}=\{G_1,\cdots, G_m\}$ be an arbitrary finite open covering of X. Set

$$X_i=\cup\{U_\alpha: \alpha\in A_i,\ \mathrm{St}(U_\alpha, \mathfrak{U}_i) \textit{ refines } \mathfrak{G}\}.$$

Then by the conditioh i) $\{X_1, X_2,\cdots\}$ is an open covering of X. Set $X_0=\phi$. Set

$$B_i=\{\alpha: \alpha\in A_i,\ U_\alpha\cap X_i\neq\phi\},$$
$$C_i=\{\alpha: \alpha\in B_i,\ U_\alpha\cap(\bigcup_{j<i} X_j)=\phi\},$$
$$D_j=\{\alpha: \alpha\in B_i,\ U_\alpha\cap(\bigcup_{j<i} X_j)\neq\phi\}.$$

Then $B_1=C_1$ and every B_i is the disjoint sum of C_i and D_i. For any i and any $\alpha\in C_i$ let

$$V_\alpha=(U_\alpha\cap X_i)\cup(\cup\{U_\beta\cap X_j: f_i^j(\beta)=a,\ \beta\in D_j,\ j>i\}).$$

Let us show that $\mathfrak{V}=\{V_\alpha: \alpha\in\bigcup_{j=1}^\infty C_i\}$ is an open covering of X such that $\mathfrak{V}$ refines $\mathfrak{G}$ and order $\mathfrak{V}\leqq n+1$, which will prove dim $X\leqq n$.

Let x be an arbitrary point of X. Then there exists i with $x\in X_i$. Take $\alpha\in B_i$ such that $x\in U_\alpha$. When $\alpha\in C_i$, then $x\in U_\alpha\cap X_i\subset V_\alpha$. When $\alpha\in D_i$, then there exists $j<i$ such that $\beta=f_j^i(\alpha)\in C_i$ since $B_1=C_1$. Then $x\in U_\alpha\cap X_i\subset V_\beta\in\mathfrak{V}$. Thus $\mathfrak{V}$ is an open covering.

Let i be an arbitrary integer and α an arbitrary index in C_i. It is clear that $U_\alpha\cap X_i\subset V_\alpha\subset U_\alpha$. There exists $\beta\in A_i$ such that $U_\beta\cap U_\alpha\cap X_i\neq\phi$ and $\mathrm{St}(U_\beta, \mathfrak{U}_i)$ refines $\mathfrak{G}$. Thus V_α refines $\mathfrak{G}$ and hence $\mathfrak{V}$ refines $\mathfrak{G}$.

To prove order $\mathfrak{V}\leqq n+1$ take an arbitrary positive integer i. Then $\{V_\alpha\cap(X_i-\bigcup_{j<i} X_j): \alpha\in\bigcup_{k=1}^\infty C_k\}=\{U_\alpha\cap X_i,\ (\cup\{U_\beta: \beta\in D_i,\ f_k^i(\beta)=\gamma\})\cap(X_i-\bigcup_{j<i} X_j): \alpha\in C_i,\ \gamma\in C_k,\ k<i\}$ and the order of the last term is at most order $\mathfrak{U}_i$. Hence order $\mathfrak{V}\wedge(X_i-\bigcup_{j<i} X_j)\leqq n+1$ and hence order $\mathfrak{V}\leqq n+1$.

Proof of Theorem 2. Suppose the condition is satisfied. Let $X_i=\{\sup_j \text{ order } (x, \mathfrak{U}_j)\leqq i\}$. Then X_i is closed and $X=\bigcup_{i=1}^\infty X_i$. If we consider the sequence of open coverings $\mathfrak{U}_j\wedge X_i$, $j=1, 2,\cdots$, of X_i, we have dim $X_i\leqq i-1$ by Theorem 3.

Proof of Theorem 4. Throughout the proof the points in K_ω are represented by their coordinates in the Hilbert cube: $K_\omega=\{(x_1,\cdots, x_i, 0, 0,\cdots): |x_j|\leqq 1/j,\ j=1,\cdots, i,\ i=1,2,\cdots\}$. Let (K_ω^*, ρ) be an arbitrary metric completion, with the metric ρ, of K_ω. If $b_1, b_2,\cdots$ are positive numbers, we set

$$J_i=\{(x_1,\cdots, x_i, 0, 0,\cdots): 0\leqq x_j\leqq b_j,\ j=1,\cdots,i\},$$
$$L_i=\{(x_1,\cdots, x_{i-1}, 0, x_{i+1},\cdots): 0\leqq x_j\leqq b_j,\ j\neq i\},$$

$$L_i'=\{(x_1,\cdots,\ x_{i-1},\ b_i,\ x_{i+1},\cdots):\ 0\leqq x_j\leqq b_j,\ j\neq i\}.$$

Then by induction we can find b_i with $0<b_i\leqq 1/i$, $i=1,2,\cdots$, which satisfy the following conditions:

i) $J_{i+1}\subset S_{1/2^i}(J_i)=\{x:\ \rho(x,\ J_i)<1/2^i\}$, $i=1,2,\cdots$.

ii) For every i and j with $j\geqq i$, $L_i\cap J_{j+1}\subset S_{a_i/2^j}(L_i\cap J_j)$ and $L_i'\cap J_{j+1}\subset S_{a_i/2^j}(L_i'\cap J_j)$, where $a_i=\rho(L_i\cap J_i,\ L_i'\cap J_i)/5$.

If we put $K=\bigcup_{j=1}^{\infty} J_i$, then K is totally bounded. Therefore $\bar{K}$ is a compact subset of K_ω^*. By our construction $\rho(L_i\cap K,\ L_i'\cap K)$ is positive for every i. Hence $\overline{L_1\cap K}$, $\overline{L_1'\cap K}$; $\overline{L_2\cap K}$, $\overline{L_2'\cap K}$; $\cdots$ is a sequence of disjoint closed pairs of $\bar{K}$. Assume that $\bar{K}$ is countable-dimensional. Then there exists a sequence of closed sets B_i of $\bar{K}$ separating $\overline{L_i\cap K}$ from $\overline{L_i'\cap K}$, $i=1,2,\cdots$, with $\bigcap_{i=1}^{\infty} B_i=\phi$. Since $\bar{K}$ is compact, there exists $n<\infty$ with $\bigcap_{i=1}^{n} B_i=\phi$. On the other hand $\bigcap_{i=1}^{n}(B_i\cap J_n)\neq\phi$, because J_n is a topological n-cell and the B_i's separate pairs of opposite faces, which is a contradiction. Therefore $\bar{K}$ is not countable-dimensional and hence K_ω^* is not countable-dimensional.

Proof of Theorem 5. By [1, D, p. 51], X has a small transfinite inductive dimension α. It is clear that α is an infinite ordinal. Now it can easily be proved by transfinite induction that for each $\beta<\alpha$ there exists a closed subset F_β of X whose small transfinite inductive dimension is β. Then F_0, F_1, $F_2,\cdots$ are what we want.

Ehime University and Duke University

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