33. *Implicit Functions on Locally Convex Topological Linear Spaces*

By Kiyoshi ISÉKI

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Some results on implicit functions on abstract spaces have been obtained by R. G. Bartle, T. H. Hildebrandt, L. M. Graves, R. Nevanlinna and G. Pulvirenti (for these bibliographies, see G. Pulvirenti [3]). In this paper, we shall investigate the results obtained by G. Pulvirenti [3] on locally convex topological linear spaces.

Let E_1, E_2, and E_3 be three locally convex topological linear spaces. We suppose that these spaces are normable by a same topological semifield R (for the concept of topological semifields, see [1]). $E_i (i=1, 2, 3)$ are locally convex topological linear spaces with the axis of R, so E_i are weakly normed spaces over the semifield R. Let $G(z)$ be a continuous linear mapping from E_3 to E_2, i.e. there is a constant element α such that $\|G(z)\| \ll \alpha \|z\|$. Then the greatest lower bound of α is called the norm of G and is denoted by $\|G\|$. Put $I_1=\{x \mid \|x-x_0\| \ll a\}$, $I_2=\{y \mid \|y-y_0\| \ll b\}$, where x_0, y_0 are fixed elements. Then the product $I_1 \times I_2$ is a subset of the product space $E_1 \times E_2$. Therefore we have the following theorem on implicit functions.

Theorem. Let $F(x, y)$ be a continuous mapping from $I_1 \times I_2$ into E_3. Let E_2 be sequential complete. If there are a continuous linear mapping $G(z)$ from E_3 into E_2 with the inverse mapping and a non-negative number α less than 1, which satisfies the following conditions

$$\text{(1)} \qquad \|F(x_0, y_0)\| < \frac{(1-\alpha)b}{\|G\|}$$

and

$$\text{(2)} \qquad \|y'-y''+G(F(x, y')-F(x, y''))\| \ll \alpha \|y'-y''\|$$

on $I_1 \times I_2$, then there are a neighborhood U of x_0 and a unique continuous mapping $y(x)$ on U such that

$$F(x, y(x))=0.$$

Proof. Consider the continuous mapping

$$f_G(x, y)=y+G(F(x, y))$$

defined on $I_1 \times I_2$ to I_2. By (2), we have

$$\text{(3)} \qquad \|f_G(x, y')-f_G(x, y'')\| \ll \alpha \|y'-y''\|$$

on $I_1 \times I_2$, therefore $f_G(x, y)$ is a continuous mapping for every x of I_1. By the condition (1) and the continuity of F, for all points of some neighborhood U of x_0,

$$\| F(x, y_0) \| \ll \frac{(1-\alpha)b}{\| G \|},$$

and, by the continuity of G,

$$\| G(F(x, y)) \| \ll \| G \| \, \| F(x, y) \|.$$

Hence we have

$$\| f_G(x, y_0) - y_0 \| \ll (1-\alpha)b.$$

We define the sequence $\{y_n(x)\}$ by

$$y_1(x) = f_G(x, y_0),$$
$$y_{n+1}(x) = f_G(x, y_n(x)). \qquad (n=1, 2, \cdots).$$

Then

$$\| y_1(x) - y_0 \| = \| f_G(x, y_0) - y_0 \| \ll (1-\alpha)b \ll b$$

on U. Suppose $\| y_n(x) - y_0 \| \ll b$, then

$$\| y_{n+1}(x) - y_0 \| \ll \| f_G(x, y_n(x)) - f_G(x, y_0) \| + \| f_G(x, y_0) - y_0 \| \ll \alpha \| y_n(x) - y_0 \| + (1-\alpha)b \ll b.$$

for $n=1, 2, \cdots$. On the other hand, since $f_G(x, y)$ is a contraction mapping, $\{y_n(x)\}$ has a unique limit function $y(x)$. Therefore $\| y(x) - y_0 \| \ll b$ on U, and by the continuity of f,

$$y(x) = f_G(x, y(x)).$$

This shows $G(F(x, y(x))) = 0$. $G(z)$ has the inverse, so we have $F(x, y(x)) = 0$. For two elements $x, x' \in U$,

$$\begin{aligned} \| y(x) - y(x') \| &= \| f_G(x, y(x)) - f_G(x, y(x')) \| \\ &\quad + \| G(F(x, y(x)) - G(F(x', y(x')) \| \\ &\ll \| f_G(x, y(x)) - f_G(x, y(x')) \| \\ &\quad + \| G \| \, \| F(x, y(x)) - F(x', y(x')) \|. \end{aligned}$$

Hence we have

$$\| y(x) - y(x') \| \ll \alpha \| y(x) - y(x') \| + \| G \| \, \| F(x, y(x)) - F(x', y(x')) \|.$$

Therefore,

$$\| y(x) - y(x') \| \ll \frac{\| G \|}{1-\alpha} \| F(x, y(x)) - F(x', y(x')) \|.$$

This show that $y(x)$ is continuous on U. This completes the proof.

If we put $F(x, y) = f(y) - x$ in Theorem, then we have the following result:

Corollary. Let $f(y)$ be a continuous mapping on I_2 to I_1. If there are a continuous linear mapping $G(x)$ on E_1 to E_2 and $0 \leq \alpha < 1$ satisfying

$$\| f(y_0) - x_0 \| \ll \frac{(1-\alpha)b}{\| G \|}$$

and

$$\| y' - y'' + G(f(y') - f(y'')) \| \ll \alpha \, \| y' - y'' \|$$

on I_2, then there are a neighborhood U of x_0 and a uuique continuous mapping $y(x)$ on U such that $f(y(x)) = x$ on U.

References

[1] M. Antonovski, V. Boltjanski, and T. Sarymsakov: Topological semifields. Tashkent (1960).

[2] G. Pulvirenti: Funzioni implicite negli spazi di Banach. Le Matematiche, **16**, 1-7 (1961).