22. *A Limit Theorem for Sums of a Certain Kind of Random Variables*

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Let $X=(X, \mathscr{B}, \mu)$ be a fixed probability space, i.e. a totally finite measure space X with a measure μ such that $\mu(X)=1$. We consider a sequence of random variables

$$\varphi_m^{(h)}(x) \quad (m=1, 2, \cdots; h\geqq 2)$$

on X which are defined by the conditions:

1) Let $\rho_1, \rho_2, \cdots, \rho_h$ be the set of h-th roots of unity. The functions $\varphi_p^{(h)}(x)$ with prime-number indices p assume the values $\rho_k (1\leqq k\leqq h)$ with equal probability $1/h$ and they are (stochastically) independent.

2) For general $m\geqq 1$ the functions $\varphi_m^{(h)}(x)$ are completely multiplicative with respect to m, i.e.

$$\varphi_{ij}^{(h)}(x)=\varphi_i^{(h)}(x)\varphi_j^{(h)}(x)$$

for any positive integers i, j: in particular $\varphi_1^{(h)}(x)=1$ with probability 1.

Apparently, the functions $\varphi_m^{(h)}(x)$ $(m=1, 2, \cdots)$ are not independent.

We write

$$s_n^{(h)}(x)=\sum_{m=1}^{n}\varphi_m^{(h)}(x) \qquad (n=1, 2, \cdots).$$

Our aim in this note is to prove the following

Theorem. *We have for any* $\varepsilon>0$

$$\lim_{n\to\infty}\frac{s_n^{(2)}(x)}{n^{\frac{1}{2}}(\log n)^{\frac{7}{4}+\varepsilon}}=0 \tag{1}$$

with probability 1 *and for* $h\geqq 3$

$$\lim_{n\to\infty}\frac{s_n^{(h)}(x)}{n^{\frac{1}{2}}(\log n)^{\frac{3}{2}+\varepsilon}}=0 \tag{2}$$

with probability 1.

According to P. Erdös (Some unsolved problems. Publ. Math. Inst. Hungar. Acad. Sci., vol. 6 ser. A (1961), pp. 221–254; especially, pp. 251–252), A. Wintner proved that for any $\varepsilon>0$ we have

$$\lim_{n\to\infty}\frac{s_n^{(2)}(x)}{n^{\frac{1}{2}+\varepsilon}}=0$$

with probability 1, and Erdös himself has improved this result to

$$\lim_{n\to\infty}\frac{s_n^{(2)}(x)}{n^{\frac{1}{2}}(\log n)^{c}}=0$$

for some constant $c>0$. We do not claim, of course, that the results stated in our theorem are the best possible of their kind. It may be conjectured that for every $h\geqq 2$ we have with probability 1

$$\limsup_{n\to\infty} \frac{\mathrm{Res}_n^{(h)}(x)}{n^{1/2}}=+\infty.$$

We note that the conjecture for $h=2$ is due to Erdös (cf. the above cited paper by him).

1. We have

$$s_n^{(h)}(x)=\sum_{j\leqq n}^{(h)}\left[\left(\frac{n}{j}\right)^{1/h}\right]\varphi_j^{(h)}(x)$$

almost everywhere on X (i.e. with probability 1), where the summation $\sum^{(h)}$ is extended over h-th power-free integers only: an integer j is said to be h-th power-free if $d^h\,|\,j$, $d>0$, implies $d=1$. For, every positive integer m can be uniquely written in the form $m=d^h j$ with some positive integers d, j, j being h-th power-free. Note that if $m=d^h j$ then $\varphi_m^{(h)}(x)=\varphi_j^{(h)}(x)$ almost everywhere on X. Also, if we denote by $\bar{\varphi}_m(x)$ the complex conjugate of $\varphi_m(x)$, then

$$\bar{\varphi}_m^{(h)}(x)=(\varphi_m^{(h)}(x))^{h-1}=\varphi_n^{(h)}(x) \qquad (n=m^{h-1})$$

almost everywhere on X. It is easy to see that the functions $\varphi_j^{(h)}(x)$ with h-th power-free indices j form an orthonormal system in X.

Lemma 1. *Let $0\leqq m<n$. Then we have*

$$\int_X \Big|\sum_{m<i\leqq n}\varphi_i^{(2)}(x)\Big|^2 d\mu=O((n^{1/2}-m^{1/2})^2\log(m+1)) \\ +O(n\log(n/(m+1)))+O(n) \qquad (n>1).$$

Proof. We have

$$\int_X \Big|\sum_{m<i\leqq n}\varphi_i^{(2)}(x)\Big|^2 d\mu=\sum_{j\leqq n}^{(2)}\left(\left[\left(\frac{n}{j}\right)^{1/2}\right]-\left[\left(\frac{m}{j}\right)^{1/2}\right]\right)^2,$$

where the sum on the right-hand side is equal to

$$\sum_{j\leqq m}^{(2)}\left(\left[\left(\frac{n}{j}\right)^{1/2}\right]-\left[\left(\frac{m}{j}\right)^{1/2}\right]\right)^2+\sum_{m<j\leqq n}^{(2)}\left[\left(\frac{n}{j}\right)^{1/2}\right]^2=\sum\nolimits_1+\sum\nolimits_2, \quad \text{say.}$$

Now

$$\sum\nolimits_1\leqq\sum_{j\leqq m}\left(\frac{n^{1/2}-m^{1/2}}{j^{1/2}}+1\right)^2 \\ =(n^{1/2}-m^{1/2})^2\log(m+1)+O(n);$$

$$\sum\nolimits_2\leqq\sum_{m<j\leqq n}\frac{n}{j}=n\log\frac{n}{m+1}+O(n).$$

This proves the lemma.

Lemma 2. *Suppose $h\geqq 3$ and let $0\leqq m<n$. Then we have*

$$\int_X \Big|\sum_{m<i\leqq n}\varphi_i^{(h)}(x)\Big|^2 d\mu=O(n).$$

Proof. The integral in the lemma equals

$$\sum_{j\leq n}^{(h)}\left(\left[\left(\frac{n}{j}\right)^{1/h}\right]-\left[\left(\frac{m}{j}\right)^{1/h}\right]\right)^2,$$

which does not exceed trivially

$$\sum_{j\leq n}\frac{n^{2/h}}{j^{2/h}}=n^{2/h}\frac{n^{1-(2/h)}}{1-(2/h)}+O(n^{2/h})=O(n).$$

We denote by $L^2(X)$, as usual, the class of measurable functions $f(x)$ defined on X such that $|f(x)|^2$ is integrable over X. Also, $\bar{f}(x)$ denotes the function conjugate to $f(x)$: thus, $f(x)\bar{f}(x)=|f(x)|^2$.

Lemma 3. *Let $f_i(x)$ $(i=1,2,\cdots)$ be a sequence of real or complex valued functions belonging to the class $L^2(X)$ and satisfying the condition*

$$\mathrm{Re}\int_X f_i(x)\bar{f}_j(x)d\mu\geqq 0$$

for any indices $i\neq j$. Then if we define

$$F_n(x)=\sup_{1\leqq m\leqq n}\left|\sum_{i=1}^{m}f_i(x)\right|,$$

we have

$$\int_X F_n^2(x)d\mu\leqq A\log^2 n\cdot\int_X\left|\sum_{i=1}^{n}f_i(x)\right|^2 d\mu \qquad (n>1)$$

with some absolute constant $A>0$.

Proof. Let $2^{r-1}<n\leqq 2^r$ $(r\geqq 1)$. We put $c_i=1$ for $1\leqq i\leqq n$, $=0$ for $n+1\leqq i\leqq 2^r$, and write for l, $0\leqq l\leqq r$,

$$F_{k,l}(x)=\left|\sum_{i=(k-1)2^{r-l}+1}^{k2^{r-l}}c_if_i(x)\right| \qquad (1\leqq k\leqq 2^l),$$

$$M_l(x)=\sup_{1\leqq k\leqq 2^l}F_{k,l}(x).$$

Considering the dyadic development of an integer m, $1\leqq m\leqq n$, we easily find that

$$F_n(x)\leqq\sum_{l=0}^{r}M_l(x),$$

and therefore

$$\int_X F_n^2(x)d\mu\leqq(r+1)\sum_{l=0}^{r}\int_X M_l^2(x)d\mu,$$

where

$$\int_X M_l^2(x)d\mu\leqq\sum_{k=1}^{2^l}\int_X F_{k,l}^2(x)d\mu$$
$$\leqq\int_X\left|\sum_{i=1}^{n}f_i(x)\right|^2 d\mu.$$

Hence

$$\int_X F_n^2(x)d\mu\leqq(r+1)^2\int_X\left|\sum_{1}^{n}f_i(x)\right|^2 d\mu.$$

Since $(r+1)^2\leqq(3/\log 2)^2\log^2 n$ $(n\geqq 2)$, our lemma is proved.

2. We are now ready to prove the theorem. First we shall demonstrate the assertion (1).

Lemma 4. *Put* $n_k=[\exp k^{\alpha}]$ $(k=1,2,\cdots)$, *where* α, $0<\alpha<1$, *is a constant. Then if* $c>(1+\alpha)/(2\alpha)$ we have

$$\lim_{k\to\infty}\frac{s_{n_k}^{(2)}(x)}{n_k^{1/2}(\log n_k)^c}=0$$

almost everywhere on X.

Proof. By Lemma 1 we have

$$\int_X |s_{n_k}^{(2)}(x)|^2\,d\mu=O(n_k\log n_k),$$

so that

$$\int_X\left(\frac{s_{n_k}^{(2)}(x)}{n_k^{1/2}(\log n_k)^c}\right)^2 d\mu=O((\log n_k)^{1-2c})=O(k^{\alpha(1-2c)}),$$

where, by assumption, $\alpha(1-2c)<-1$. Hence

$$\sum_{k=1}^{\infty}\int_X\left(\frac{s_{n_k}^{(2)}(x)}{n_k^{1/2}(\log n_k)^c}\right)^2 d\mu<\infty,$$

and the series

$$\sum_{k=1}^{\infty}\left(\frac{s_{n_k}^{(2)}(x)}{n_k^{1/2}(\log n_k)^c}\right)^2$$

converges almost everywhere on X. The result follows from this at once.

Let $h=2$ and put again

$$n_k=[\exp k^{\alpha}]\qquad(k=1,2,\cdots),$$

where α, $0<\alpha<1$, will be determined in a moment later. Define

$$G_k(x)=\sup_{n_k<n\leqq n_{k+1}}|s_n^{(2)}(x)-s_{n_k}^{(2)}(x)|\qquad(k=1,2,\cdots).$$

Since we have $\int\varphi_i^{(2)}(x)\varphi_j^{(2)}(x)d\mu\geqq 0$ for any indices i, j, Lemma 3 is applicable to $G_k(x)$.

We see that $n_{k+1}-n_k=O(k^{\alpha-1}n_k)=o(n_k)$, $(n_{k+1}^{1/2}-n_k^{1/2})^2=O(k^{2\alpha-2}n_k)$, so that by Lemma 1,

$$\int_X\left|\sum_{n_k<i\leqq n_{k+1}}\varphi_i^{(2)}(x)\right|^2 d\mu=O(k^{2\alpha-2}n_k\log n_k)+O(n_k).$$

It now follows from Lemma 3 that

$$\int_X G_k^2(x)d\mu=O(k^{2\alpha-2}n_k\log^3 n_k)+O(n_k\log^2 n_k).$$

Hence if $c>0$ is a constant then

$$\int_X\left(\frac{G_k(x)}{n_k^{1/2}(\log n_k)^c}\right)^2 d\mu=O(k^{(5-2c)\alpha-2})+O(k^{(2-2c)\alpha}),$$

since $\log n_k=k^{\alpha}+o(1)$. Choose $\alpha=2/3$ and suppose $c>7/4$. Then $(5-2c)\alpha-2=(2-2c)\alpha<-1$, and we have

$$\sum_{k=1}^{\infty}\int_X\left(\frac{G_k(x)}{n_k^{1/2}(\log n_k)^c}\right)^2 d\mu<\infty,$$

from which we deduce, as in the proof of Lemma 4, that

$$\lim_{k\to\infty}\frac{G_k(x)}{n_k^{1/2}(\log n_k)^c}=0$$

almost everywhere on X. Applying Lemma 4 with $\alpha=2/3$, we thus conclude finally that

$$\lim_{n\to\infty}\frac{s_n^{(2)}(x)}{n^{1/2}(\log n)^c}=0$$

almost everywhere on X, provided that $c>7/4$. This proves (1).

The proof of (2) is similar to that of (1), but the argument is somewhat simpler.

Lemma 5. *Suppose $h\geqq 3$ and put $n_k=[e^k]$ $(k=1,2,\cdots)$. Then if $c>1/2$ we have*

$$\lim_{k\to\infty}\frac{s_{n_k}^{(h)}(x)}{n_k^{1/2}(\log n_k)^c}=0$$

almost everywhere on X.

Proof. The result follows easily from Lemma 2.

Now define for $k=1,2,\cdots$

$$H_k(x)=\sup_{n_k<n\leqq n_{k+1}}|s_n^{(h)}(x)-s_{n_k}^{(h)}(x)|,$$

where $n_k=[e^k]$. It is clear that Lemma 3 is also applicable to $H_k(x)$. By Lemmas 2 and 3 we obtain

$$\int_X H_k^2(x)d\mu=O(n_k\log^2 n_k),$$

and therefore, if $c>0$ is a constant then

$$\int_X\left(\frac{H_k(x)}{n_k^{1/2}(\log n_k)^c}\right)^2 d\mu=O(k^{2-2c}).$$

Thus, arguing just as before, we find that

$$\lim_{k\to\infty}\frac{H_k(x)}{n_k^{1/2}(\log n_k)^c}=0$$

almost everywhere on X, if $c>3/2$. This together with Lemma 5 implies (2).

Our proof of the theorem is now complete.

Remark. It will be clear from the proof that the denominators in the left-hand side of (1) and (2) may be replaced respectively by

$$n^{\frac{1}{2}}(\log n)^{\frac{7}{4}}(\log\log n)^{\frac{1}{2}+\varepsilon}$$

and

$$n^{\frac{1}{2}}(\log n)^{\frac{3}{2}}(\log\log n)^{\frac{1}{2}+\varepsilon}$$

for any $\varepsilon>0$, without affecting the results.