21. *A Certain System of Parameters in a Local Ring*

By Yasuo KINUGASA
Department of Mathematics, Tokyo Metropolitan University
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Let R be a (noetherian) local ring with maximal ideal $\mathfrak{M}$: denoted by $(R, \mathfrak{M})$. First we set the following

Definition. In a local ring $(R, \mathfrak{M})$ we call a system of parameters $a_1, \cdots, a_r$ of R satisfying the conditions; $a_i \notin \mathfrak{M}^2 + \sum_{j<i} a_j R \ (1 \leq i \leq r)$ a *special system of parameters* of R, where $r = \text{Alt.}\, R$[1] (altitude of R = Krull dimension of R).

In this note, by using this notion of a special system of parameters, we shall prove the following:

Theorem. *In a local ring $(R, \mathfrak{M})$ the following three conditions are equivalent to each other*:

(1) *R is a Macaulay local ring.*

(2) *If $a_1, \cdots, a_r$ is a system of parameters of R, then $hd_R \sum_{i=1}^{r} a_i R$*[1] *$< \infty$.*

(3) *There exists a special system of parameters $a_1, \cdots, a_r$ such that $hd_R \sum_{i=1}^{r} a_i R < \infty$.*

For the proof of the theorem we need the following lemmas.

Lemma 1. *Let $\mathfrak{A}$ be an ideal of a local ring $(R, \mathfrak{M})$ and $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be prime ideals of R. If $\mathfrak{B}$ is an ideal of R such that $\mathfrak{B} \not\subseteq \mathfrak{A}$ and $\mathfrak{B} \not\subseteq \bigcup_{j=1}^{n} \mathfrak{P}$, then $\mathfrak{B} \not\subseteq \mathfrak{A} \cup \mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_n$.*

Proof. See [2, p. 70. Prop. 2].

Lemma 2. In a local ring $(R, \mathfrak{M})$ there exists a special system of parameters.

Proof. We shall show how to construct inductively such a system of parameters. It is obvious if $\text{Alt.}\, R = 0$. Let $r = \text{Alt.}\, R \geq 1$ and let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be the minimal prime divisors of zero. Take a_1 such that $a_1 \in \mathfrak{M}$, $a_1 \notin \mathfrak{M}^2$ and $a_1 \notin \mathfrak{P}_i \ (i=1, \cdots, n)$ by Lemma 1. Then the height of $a_1 R$ is one. After choosing $a_i, \cdots, a_t \ (t<r)$, we can take a_{t+1} in M such that $a_{t+1} \notin \mathfrak{M}^2 + \sum_{j=1}^{t} a_i R$ and $a_{t+1} \notin \mathfrak{Q}_i \ (i=1, \cdots, m)$ by Lemma 1, where $\mathfrak{Q}_j$'s are the minimal prime divisors of $\sum_{j=1}^{t} a_j R$. It is obvious that the height of $\sum_{j=1}^{t+1} a_j R$ is $t+1$. Thus, we obtain a special system of parameters $a_1, \cdots, a_r$ of R.

Remark. If $a_1, \cdots, a_r$ is a special system of parameters of R

1) Concerning notations see [3].

then $a_1, \cdots, a_{t-1}, a_{t+1}, a_t, a_{t+2}, \cdots, a_r$ is also a special system of parameters of R. Consequently, if $a_1, \cdots, a_r$ is a special system of parameters of R, then $a_{\sigma_1}, \cdots, a_{\sigma_r}$ is also a special system of parameters of R for any parmutation σ of the set $\{1, 2, \cdots, r\}$.

Lemma 3. *Let $a_1, \cdots, a_r$ be a special system of parameters of a local ring $(R, \mathfrak{M})$ and $\mathfrak{Q}=\sum_{j=1}^{r} a_jR$. If $0:\mathfrak{M}=0$, then there exists a special system of parameters $a_1', \cdots, a_r'$, with a_1' as a non-zero-divisor in R, and such that $\mathfrak{Q}=\sum_i a_i'R$.*

Proof. $0:\mathfrak{M}=0$ implies $0:\mathfrak{Q}=0$ since $\mathfrak{Q}$ is an $\mathfrak{M}$-primary ideal. Then we have b in $\mathfrak{Q}$ such that $b=s_1a_1+\cdots+s_ra_r\ (s_j \in R)$, $b \notin \mathfrak{M}^2$ and b is a non zero divisor in R by Lemma 1. We may assume $s_r \notin \mathfrak{M}$ since $b \notin \mathfrak{M}^2$ and $a_1, \cdots, a_{j-1}, a_r, a_{j+1}, \cdots, a_{r-1}$ is a special system of parameters of R (Remark to Lemma 2). It is obvious that $\mathfrak{Q}=\sum_{j=1}^{r-1} a_jR+bR$. If $b=s_1a_1+\cdots+s_ra_r \in \mathfrak{M}^2+\sum_{j=1}^{r-1} a_jR$ then $s_ra_r \in \mathfrak{M}^2+\sum_{j=1}^{r-1} a_jR$, hence we have $a_r \in \mathfrak{M}^2+\sum_{j=1}^{r-1} a_jR$ since s_r is a unit in R. This contradicts to the condition $a_r \notin \mathfrak{M}^2+\sum_{j=1}^{r-1} a_jR$. Thus we have a special system of parameters $a_1, \cdots, a_{r-1}, b$ of R. Put $a_1'=b$, $a_i'=a_i\ (i=2, \cdots, r-1)$ and $a_r'=a_1$, then $a_1', \cdots, a_r'$ is a special system of parameters of R by Remark to Lemma 2.

Lemma 4. *In a local ring $(R, \mathfrak{M})$, if $0:\mathfrak{M}\neq 0$, then $hd_RM=\infty$ for every finite R-module M which is not free.*

Proof. See [3, (28.1)].

Lemma 5. *Let $\mathfrak{Q}$ be an ideal of a local ring $(R, \mathfrak{M})$ and x be an element of $\mathfrak{Q}$ which is not contained in $\mathfrak{M}^2$. Assume furthermore that x is not a zero-divisor in R. Then we have $hd_R\mathfrak{Q}=1+hd_{R/xR}(\mathfrak{Q}/xR)$.*

Proof. See [3, (27.4)].

Lemma 6. *Let $(R, \mathfrak{M})$ be a local ring. If $x_1, \cdots, x_d$ is an R-sequence $(x_j \in \mathfrak{M})$, then $hd_R \sum_{j=1}^{d} x_jR<\infty$.*

Proof. We shall prove this lemma by induction on d. If $d \leq 1$ the proof is obvious. Let $d>1$ and $\mathrm{syz}_R^1\left(\sum_{j=1}^{d} x_jR\right)=\left\{\sum_{j=1}^{d} c_jX_j \,\middle|\, \sum_{j=1}^{d} c_jx_j=0, c_j \in R \text{ and } X_j\text{'s are indeterminates}\right\}$. Furthermore define R-homomorphism φ: $\mathrm{syz}_R^1\left(\sum_{j=1}^{d} x_jR\right)\to R$ by $\varphi\left(\sum_{j=1}^{d} c_jX_j\right)=c_d$. We have immediately Image $\varphi=\sum_{j=1}^{d-1} x_jR$ and Kernel $\varphi=\mathrm{syz}_R^1\left(\sum_{j=1}^{d-1} x_jR\right)$ since $\left(\sum_{j=1}^{d-1} x_jR\right): x_dR=\left(\sum_{j=1}^{d-1} x_jR\right)$. Thus we have the following exact sequence:

$$0\to \mathrm{syz}_R^1\left(\sum_{j=1}^{d-1} x_jR\right)\to \mathrm{syz}_R^1\left(\sum_{j=1}^{d} x_jR\right)\to \sum_{j=1}^{d-1} x_jR\to 0.$$

By our inductive assumption we have $hd_R \sum_{j=1}^{d-1} x_jR<\infty$ and

$$hd_R\left(\mathrm{syz}_R^1\left(\sum_{j=1}^{d-1} x_jR\right)\right)<\infty\ .$$

Hence we have $hd_R\left(\mathrm{syz}_R^1\left(\sum_{j=1}^{d} x_jR\right)\right)<\infty$ (See [3], (26.6)). This implies $hd_R \sum_{j=1}^{d} x_jR<\infty$.

Proof of the theorem

(1)⇒(2): The proof follows from Lemma 6.

(2)⇒(3): The proof is obvious.

(3)⇒(1): We shall complete the proof by using induction on the altitude of R. If $\mathrm{Alt}.R=0$, R is always a Macaulay local ring. Let $r=\mathrm{Alt}.R\geq 1$ and $\mathfrak{Q}=\sum_{j=1}^{r} a_jR$. If $0:\mathfrak{M}\neq 0$, by Lemma 4 $\mathfrak{Q}$ is a free module since $hd_R\mathfrak{Q}<\infty$. On the other hand we have $(0:\mathfrak{M})\mathfrak{Q}=0$ and $0:\mathfrak{M}\neq 0$. Hence we conclude $\mathfrak{Q}=0$, which contradicts to the assumption. Thus we have $0:\mathfrak{M}=0$. By the fact $0:\mathfrak{M}=0$ we may assume that a_1 is a non zero divisor in R (Lemma 3). It is obvious that the set of a_2 modulo $a_1R, \cdots, a_r$ modulo a_1R is a special system of parameters of R/a_1R since $a_1, \cdots, a_r$ is a special system of parameters of R. On the other hand we have $hd_R\mathfrak{Q}=1+hd_{R/a_1R}(\mathfrak{Q}/a_1R)$ by Lemma 5 and $\mathrm{Alt}.R=\mathrm{Alt}.R/a_1R+1$. Hence R/a_1R is a Macaulay local ring by the assumption of induction. This implies that R is a Macaulay local ring since a_1 is a non-zero-divisor in R.

Appendix

By using the notion of the special system of parameters we have the Theorem (Cf. Theorem of [4]). *In a local ring* $(R, \mathfrak{M})$, *the following conditions are equivalent*:

(1) *R is a Gorenstein local ring.*[2]

(2) *There exists an $\mathfrak{M}$-primary ideal $\mathfrak{Q}$ generated by a special system of parameters satisfying the following*:

For any special system of parameters $a_1, \cdots, a_r$ which generates $\mathfrak{Q}$, all $\mathfrak{Q}_{(n,i)}$ are irreducille, where $\mathfrak{Q}_{(n,i)}=\sum_{j=1}^{i-1} a_jR+\sum_{j=i}^{r} a_j^nR$ $(r=\mathrm{Alt}.R, i=1, \cdots, r$ *and* $n=1, 2, \cdots)$.

Proof. (1)⇒(2) is obvious. For the proof of implication (2)⇒(1) it will be sufficient only to prove that R is a Macaulay local ring. We shall prove it by induction on the altitude of R. If $\mathrm{Alt}.R=0$, the proof is obvious. If $\mathrm{Alt}.R\geq 1$, each $\mathfrak{Q}_{(n,1)}=\sum_{j=1}^{r} a_j^nR$ is irreducible and $\mathfrak{Q}_{(n,1)}\subsetneqq\mathfrak{Q}_{(n-1,1)}$ $(n\geq 2)$. So we have $\mathfrak{Q}_{(n,1)}:\mathfrak{M}\subseteq\mathfrak{Q}_{(n-1,1)}$[3] $(n\geq 2)$. While we have $\bigcap_{n=1}^{\infty}\mathfrak{Q}_{(n,1)}\subseteq\bigcap_{n=1}^{\infty}\mathfrak{M}^n=0$, therefore $0:\mathfrak{M}=\left(\bigcap_{n=2}^{\infty}\mathfrak{Q}_{(n,1)}\right):\mathfrak{M}=\bigcap_{n=2}^{\infty}(\mathfrak{Q}_{(n,1)}:\mathfrak{M})\subseteq\bigcap_{n=2}^{\infty}\mathfrak{Q}_{(n-1,1)}=0$. Since $0:\mathfrak{M}=0$, we have a special system

2) See [1].

3) See [5, p. 248, Th. 34].

of parameters $a_1, \cdots, a_r$ with a_1 as a non zero divisor which generates $\mathfrak{Q}$ by Lemma 3. For any special system of parameters $\bar{b}_2, \cdots, \bar{b}_r$ of R/a_1R, which generates $\mathfrak{Q}/a_1R$, $a_1, b_2, \cdots, b_r$ is a special system of parameters of R which generates $\mathfrak{Q}$, where $\bar{b}_j = b_j$ modulo a_1R. Let $\mathfrak{Q}'_{(n,i)}$ be the ideal of R/a_1R generated by $\bar{b}_2, \cdots, \bar{b}_{i-1}, \bar{b}_i^n, \cdots, \bar{b}_r^n$ $(i=2, \cdots, r, n=1, 2, \cdots)$. Then all $\mathfrak{Q}'_{(n,i)}$ are irreducible since all $a_1R+b_2R+\cdots+b_{i-1}R+b_i^nR+\cdots+b_r^nR$ are irreducible by our assumption. Hence R/a_1R is a Macaulay local ring by our inductive assumption since Alt. R = Alt. R/a_1R+1. This implies that R is a Macaulay local ring.

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