51. Inversive Semigroups. III

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§ 1. **Introduction.** This paper is the continuation of the previous papers [8] and [9].

A semigroup G is said to be *regular* if it satisfies the following condition:

(C) For any element a of G, there exists an element x such that $axa=a$.

For example, inversive semigroups introduced by [8] are regular.

Now, consider the identity

(P) $x_1x_2x_3 \cdots x_n = x_{p_1}x_{p_2}x_{p_3} \cdots x_{p_n}$,

where $(p_1, p_2, p_3, \cdots, p_n)$ is a non-trivial permutation of $(1, 2, 3, \cdots, n)$.

Such an identity is called a *permutation identity*. If (P) is valid for any elements $x_1, x_2, x_3, \cdots, x_n$ of a semigroup M, then we shall say that M *satisfies* the permutation identity (P). For example, *commutativity* $x_1x_2=x_2x_1$ and *normality* $x_1x_2x_3x_4=x_1x_3x_2x_4$ are clearly permutation identities. A semigroup satisfying commutativity $x_1x_2=x_2x_1$ is usually called a commutative semigroup. Similarly, we shall say that a semigroup is *normal* if it satisfies normality $x_1x_2x_3x_4=x_1x_3x_2x_4$. It is clear that any group satisfying a permutation identity is commutative. Further, we shall show later that any inverse semigroup introduced by Vagner [5] under the name "generalized group" is commutative if it satisfies a permutation identity. However, a regular semigroup satisfying a permutation identity is not necessarily commutative, and is sometimes quite different from commutative semigroups. This is easily seen from the fact that a rectangular band R is a regular semigroup satisfying normality, but any two different elements of R do not commute.[1] Special kinds of regular semigroups satisfying permutation identities have been studied by many papers (e.g., Clifford [1], Preston [3], Clifford & Preston [2], Thierrin [4], the author [6], [8], [9] and Kimura & the author [10]). Especially, Clifford [1] and the author [7] completely determined the structure of commutative inversive semigroups and gave an explicit description of a method of constructing all possible commutative inversive semigroups. On the other hand, Kimura & the author [10] clarified the structure of bands satisfying various

1) See Clifford and Preston [2], p. 26.

permutation identities. In particular, it was proved that any band satisfying a permutation identity satisfies normality. And a structure theorem for normal bands was also given by [10]. Further, the author [8] recently clarified the structure of normal inversive semigroups by use of the structure theorem for normal bands:

Structure theorem for normal inversive semigroups. A semigroup G is isomorphic to the spined product of a commutative inversive semigroup A and a normal band B if and only if it is normal and inversive. Further, in this case the set of all idempotents of G is a subband of G and isomorphic to B.[2]

The main purpose of this paper is to present a theorem which shows that any regular semigroup satisfying a permutation identity is necessarily a normal inversive semigroup. Also, one further theorem will be given to clarify the relation between several conditions on semigroups. All notations and terminology should be refered to [8] and [10]. The proofs are almost omitted and will be given in detail elsewhere.

§2. **Regular semigroups satisfying permutation identities.** Let S be a regular semigroup satisfying the permutation identity (P). Since S is regular, the set of all idempotents of S is not empty. We shall denote it by I.

Then, we can easily prove the following lemmas by simple calculation.

Lemma 1. *I is a normal subband of S.*

Lemma 2. *S is an inversive semigroup.*

Since S is inversive and I is a normal band, by Corollary 1 of [8] S is (N)-inversive and isomorphic to the spined product of a (C)-inversive semigroup C and the normal band $I: C \infty I(\Gamma) \cong S$. Let $C \sim \sum\{C_\gamma : \gamma \in \Gamma\}$, $I \sim \sum\{I_\gamma : \gamma \in \Gamma\}$ be the structure decompositions of C and I respectively. Then, it is easily seen that each C_γ and each I_γ are a group and a rectangular band respectively. And the structure decomposition of $C \infty I\ (\Gamma)$ is $C \infty I\ (\Gamma) \sim \sum\{C_\gamma \times I_\gamma : \gamma \in \Gamma\}$. Since S satisfies (P), each C_γ also satisfies (P). Therefore, each C_γ is a group satisfying (P), and hence C_γ is commutative.

Further, this result can be extended to the following lemma.

Lemma 3. *C is a commutative inversive semigroup.*

From the above-mentioned results, S is isomorphic to the spined product of a commutative inversive semigroup and a normal band. Hence, by the structure theorem for normal inversive semigroups the semigroup S is normal and inversive. Conversely, it is obvious that any normal inversive semigroup is a regular semigroup satisfying

2) For the definition of the spined product of semigroups, see the author [8].

the permutation identity $x_1x_2x_3x_4=x_1x_3x_2x_4$.

Therefore, we have

Theorem 1. *A regular semigroup satisfying a permutation identity is isomorphic to the spined product of a commutative inversive semigroup and a normal band. Accordingly, it is a normal inversive semigroup. Conversely, any normal inversive semigroup is a regular semigroup satisfying a permutation identity.*

Let P_1 and P_2 be two permutation identities. We shall say that *P_1 implies P_2 on semigroups of type T* if every semigroup having type T satisfies P_2 whenever it satisfies P_1. In [10] the author has shown that any permutation identity implies normality on bands. This result can be extended to the following corollary to Theorem 1.

Corollary. *Any permutation identity implies normality $x_1x_2x_3x_4=x_1x_3x_2x_4$ on regular semigroups.*

Finally, we shall show one further theorem which can be proved by use of the above-mentioned theorem, the paper [8] of the author and Theorem 4.3 of Clifford & Preston [2].

Theorem 2. *For a semigroup M satisfying a permutation identity, the following conditions are equivalent:*

(1) *M is regular.*

(2) *M is left regular and right regular.*

(3) *M is inversive.*

(4) *M is strictly inversive.*

(5) *M is (N)-inversive.*

(6) *M is normal and inversive.*

(7) *M is the union of (commutative) groups.*

(8) *M is a semilattice of (R)-inversive semigroups.*

(9) *M* is *a (normal) band of (commutative) groups.*

(10) *M is isomorphic to the spined product of a (normal) band and a (C)-inversive semigroup.*

(11) *M is isomorphic to the spined product of a (normal) band and a commutative inversive semigroup.*

Let K be an inverse semigroup satisfying a permutation identity. Since K is of course regular, by (11) of Theorem 2 the semigroup K is isomorphic to the spined product of a commutative inversive semigroup A and a band $B : K \cong A \infty B\ (\Omega)$. Since the set of idempotents of K is commutative, B is also commutative. Therefore, $B \cong \Omega$, and hence $K \cong A$. Thus, K is commutative.

Consequently, we have

Corollary. *An inverse semigroup satisfying a permutation identity is commutative.*

References

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