48. On the Ranges of the Increasing Mappings

By Sadayuki YAMAMURO*)

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Let E be a real Banach space, G be an open subset and $\bar{G}$ be the closure of G. In [3] (cf. [4] and [5]), we gave the following definitions:

A mapping $f: \bar{G} \to E$ *is said to be* (δ_0)*-increasing at* $a \in G$ *if* f *satisfies the following two conditions:*

1°. $\|x\| < \delta_0$ *implies* $a + x \in G$;

2°. $f(a+x) - f(a) \neq \alpha x$ *if* $\alpha \leqq 0$ *and* $0 < \|x\| < \delta_0$.

A mapping $f: \bar{G} \to E$ *is said to be* $(\varepsilon_0, \delta_0)$*-uniformly increasing at* $a \in G$ *if* f *satisfies the following conditions:*

1°. $\|x\| < \delta_0$ *implies* $a + x \in G$;

3°. $\|f(a+x) - f(a) - \alpha x\| \geqq \varepsilon_0 \|x\|$ *if* $\alpha \leqq 0$ *and* $0 < \|x\| < \delta_0$.

It is evident that, if a mapping $f: \bar{G} \to E$ is $(\varepsilon_0, \delta_0)$-uniformly increasing at a, then f is (δ_0)-increasing at a.

The following two facts immediately follow from the above definitions.

Theorem 1. *If a mapping* $f: E \to E$ *is* (∞)*-increasing at every point of* E, *then* f *is one-to-one.*

Theorem 2. *If a mapping* $f: E \to E$ *is* (ε_0, ∞)*-uniformly increasing at every point of* E, *then, for any non-positive number* α, *the range of* $f(x) - \alpha x$ *is closed.*

A mapping $f: \bar{G} \to E$ is said to be *a completely continuous vector field on* $\bar{G}$ if f is continuous on $\bar{G}$ and the image $F(\bar{G})$ by the mapping $F(x) = x - f(x)$ is contained in a compact set. We shall say that f is a completely continuous vector field *on* E if it is a completely continuous vector field on any closed ball $\bar{B}(r) = \{x \in E \mid \|x\| \leqq r\}$.

Then, we can prove the following

Theorem 3. *Let* $f: E \to E$ *be a mapping. Suppose that*

4°. f *is* (ε_0, ∞)*-uniformly increasing at every point of* E;

5°. f *is a completely continuous vector field on* E.

Then, the mapping f *is onto, one-to-one and bicontinuous.*

Proof. Theorem 1 and the condition 4° imply that f is one-to-one. Theorem 2 and the condition 4° imply that $f(E)$ is closed. We have only to prove that $f(E)$ is open.

Assume that $y_0 \in f(E)$, namely, $y_0 = f(x_0)$ for some $x_0 \in E$. There

*) Department of Mathematics, Institute of Advanced Studies, Australian National University.

exists an open ball $B(r)=\{x\in E \mid \|x\|<r\}$ such that $x_0\in B(r)$. By the condition 5°, f is a completely continuous vector field on the closed ball $\bar{B}(r)$. Therefore, Theorem 3 of [3] implies that $f[B(r)]$ is an open set in E, and, evidently, $y_0\in f[B(r)]\subset f(E)$, which means that $f(E)$ is open.

Therefore, f should be onto. The continuity of the inverse mapping of f follows from the condition 3° if we take $\alpha=0$.

Next, we consider the case when the mapping f is not a completely continuous vector field but is Fréchet-differentiable on E.

A mapping $f\colon \bar{G}\to E$ is said to be *Fréchet-differentiable at* $a\in G$ if the exists a continuous linear mapping $D\colon E\to E$ such that

6°. $$f(a+x)-f(a)=D(x)+o(\|x\|),$$

where $o(\|x\|)/\|x\|\to 0$ if $\|x\|\to 0$. We denote this continuous linear mapping D by $f'(a)$ and call *the Fréchet-derivative of f at a.*

Then, we prove the following

Theorem 4. *Let $f\colon E\to E$ be Fréchet-differentiable at every point of E. Then, f is $(\varepsilon_0, \delta_0)$-uniformly increasing at a if and only if the Fréchet-derivative $f'(a)$ of f at a is (ε_1, ∞)-uniformly increasing at* 0 *for some $\varepsilon_1>0$.*

Proof. Let f be $(\varepsilon_0, \delta_0)$-uniformly increasing at a. Then, from 6° it follows that, if $\alpha\leqq 0$ and $\|x\|<\delta_0$,

$$\begin{aligned}\|f'(a)(x)-\alpha x\| &= \|f(a+x)-f(a)-\alpha x-o(\|x\|)\| \\ &\geqq \|f(a+x)-f(a)-ax\|-\|o(\|x\|)\| \\ &\geqq \varepsilon_0\|x\|-\|o(\|x\|)\|.\end{aligned}$$

There exists $\delta_1>0$ such that $\delta_1\leqq\delta_0$ and $\|o(\|x\|)\|<\frac{1}{2}\varepsilon_0\|x\|$ if $\|x\|<\delta_1$. Therefore, if $\|x\|<\delta_1$ and $\alpha\leqq 0$, we have

$$\|f'(a)(x)-\alpha x\|\geqq \tfrac{1}{2}\varepsilon_0\|x\|,$$

which means that $f'(a)$ is $(\frac{1}{2}\varepsilon_0, \delta_1)$-uniformly increasing at 0, and, since $f'(a)$ is a linear mapping, δ_1 can be replaced by ∞.

The converse of this theorem can be proved similarly.

Theorem 5. *Let $f\colon E\to E$ be a mapping. Suppose that*

4°. *f is (ε_0, ∞)-uniformly increasing at every point of E;*

7°. *f is Fréchet-differentiable at every point of E and the Fréchet-derivative $f'(a)$ is continuous with respect to a;*

8°. *$f'(a)(E)=E$.*

Then, the mapping f is onto, one-to-one and bicontinuous.

Proof. The facts that f is one-to-one and $f(E)$ is closed can be proved in the same way as in the proof of Theorem 3. We have only to prove that $f(E)$ is open. For this purpose, we shall use the Implicit Function Theorem (for example, [1], p. 12, Theorem 1), which insures that, if $f'(a)$ is onto, one-to-one and bicontinuous, f is an open mapping. By Theorem 4, $f'(a)$ is one-to-one and bicontinuous, but it is not necessarily onto. This is the reason why we need the condition 8°.

Remark. Let E be a real Hilbert space, $\phi(x)$ be a real-valued functional on E and a mapping $f(x)$ be the Fréchet-derivative of $\phi(x)$. If the mapping $f(x)$ is Fréchet-differentiable at every point of E, then the mapping $f'(a)$ is a symmetric operator ([2], p. 56, Theorem 5.1.). Therefore, when E is a real Hilbert space, the conclusion of Theorem 5 remains true if the condition 8° is replaced by the following condition: *$f(x)$ is a continuous Fréchet-derivative of a functional on E.*

References

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