62. On Linear Holonomy Group of Riemannian Symmetric Spaces

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Let M be a connected Riemannian manifold with Riemannian structure g, of dimension n and of class C^∞, and let M_p be the tangent space of M at p. We denote by L_p the group of all linear transformations of M_p. Let A_p be the subgroup of L_p consisting of all elements of L_p which leave invariant the scalar product $g_p(X, Y)$, the curvature tensor R and its successive covariant differentials $\nabla^k R,(k=1, 2, \cdots)$ at p. A_p is a Lie group as a closed subgroup of the Lie group L_p. We denote by $h(p)$ the linear holonomy group of M at p. $h(p)$ is a Lie group, and it's identity component $h(p)^0$ is the restricted linear holonomy group of M at p [3]. In this note we shall denote by G^0 the identity component of a Lie group G.

Theorem 1. *Let M be a Riemannian locally symmetric space, then the restricted holonomy group $h(p)^0$ is contained in A_p^0 at each point p in M.*

Proof. Since M is an analytic Riemannian manifold, the Lie algebra of $h(p)$ consists of the following matrix [3],

$$\sum_{r\,s} \lambda_{rs} R_{rs} \quad \text{where } (R_{rs})_{ij} = (R_{ijrs})_p \,.$$

We take a local coordinate system $(x_1, \cdots, x_n)$ at p such that $\{(\partial/\partial x_1)_p, \cdots, (\partial/\partial x_n)_p\}$ is an orthonormal base of M_p. We express each element of A_p by a matrix with respect to the above base. Then A_p consists of all orthogonal matrices $\| a_{ij} \|$ which satisfy

$$\sum_{\alpha\,\beta\,\gamma\,\delta} a_{i\alpha} a_{j\beta} a_{k\gamma} a_{l\delta} (R_{\alpha\beta\gamma\delta})_p = (R_{ijkl})_p \,.$$

Therefore the Lie algebra of A_p consists of all skew symmetric matrices $\| \mu_{ij} \|$ which satisfy

$$\{\mu_{ih}(R_{hjkl})_p + \mu_{jh}(R_{ihkl})_p + \mu_{kh}(R_{ijhl})_p + \mu_{lh}(R_{ijkh})_p\} = 0.$$

From the Ricci identity we have

$$\nabla_s \nabla_r R_{ijkl} - \nabla_r \nabla_s R_{ijkl} = - \sum_h \{R^h_{irs} R_{hjkl} + R^h_{jrs} R_{ihkl} + R^h_{krs} R_{ijhl} + R^h_{lrs} R_{ijkl}\} \,.$$

Since M is locally symmetric, the left sides of this expression vanish. By lowering the index h and making use of the identities $R_{ijrs} = -R_{ijrs}$,

$$\sum_h \{(R_{ihrs})_p (R_{hjkl})_p + (R_{jkrs})_p (R_{ihkl})_p + (R_{khrs})_p (R_{ijhl})_p + (R_{lhrs})_p (R_{ijkh})_p\} = 0 \,.$$

This means that the Lie algebra af $h(p)$ is contained in the Lie algebra of A_p. Therefore we have $h(p)^0 \subset A_p^0$.

We denote by $I(M)$ the group of isometries of M, and by H_p the isotropy group of $I(M)$ at p, and by dH_p the linear isotropy group of H_p. We have proved in [5] that for a simply connected Riemannian globally symmetric space M, $A_p = dH_p$ at each p in M. Since M is simply connected $h(p)^0 = h(p)$. Therefore we have the following.

Corollary. *Let M be a simply connected Riemannian globally symmetric space, then $h(p) \subset dH_p$.*

Theorem 2. *Let M be a simply connected compact Riemannian globally symmetric space, then $h(p) = A_p^0$.*

In order to prove Theorem 2 we need some lemmas. We denote by J_p the isotropy group of $I(M)$ at p, and by dJ_p the linear isotropy group of J_p.

Lemma 1. *Let M be a Riemannian globally symmetric space,* then $H_p^0 = J_p$.

Proof. Since M is a Riemannian globally symmetric space, $I(M)^0$ acts transitively on M[2]. We choose a subset Q of $I(M)$ such that $I(M) = \bigcup_{a \in Q} aI(M)^0$ and $aI(M)^0 \cap bI(M)^0 = \emptyset$ whenever $a \neq b (a, b \in Q)$. Since $I(M)^0 = \bigcup_{p \in M} J_p$, we have

$$I(M) = \bigcup_{a \in Q} (\bigcup_{p \in M} aJ) = \bigcup_{p \in M} (\bigcup_{a \in Q} aJ) .$$

But $I(M) = \bigcup_{p \in M} H_p$, and $aJ_p \cap bJ_p = \emptyset$ whenever $a \neq b (a, b \in Q)$. Therefore we have $H_p^0 = J_p$.

Lemma 2. *Let M be a simply connected Riemannian globally symmetric space, then $(dH_p)^0 = dJ_p$.*

Proof. We have proved in [5] that for a simply connected analytic complete Riemannian manifold, H_p is isomorphic to dH_p as Lie groups, and that this isomorphism is given by the correspondence $f \in H_p \to (df)_p \in dH_p$. Therefore $(dH_p)^0$ coincides with $d(H_p^0)$ which is the image of H_p^0 under this isomorphism. By Lemma 1 $d(H_p^0)$ coincides with dJ_p.

Proof of Theorem 2. Since M is a simply connected Riemannian globally symmetric space, we have $A_p = dH_p$[5]. Therefore by lemma 2 we get $A_p^0 = dJ_p$. Since M is compact, dJ_p is contained in $h(p)$ [6], and hence $A_p^0 \subset h(p)$. On the other hand, from Theorem 1 $h(p)^0 \subset A_p^0$. Since M is simply connected, we have $h(p)^0 = h(p)$.

Example. Consider in E^{n+1} a sphere $S^n (n \geqq 2)$ with the natural Riemannian metric. S^n satisfies the conditions of Theorem 2. Since S^n is of constant curvature, $A_p = 0(n)$, the rotation group of E^n. Therefore we have $h(p) = 0(n)^0$.

Theorem 3. *If M is a Riemannian globally symmetric space, and for some positive number α the Ricci curvature K satisfies*

$K(U, U) \geqq \alpha$ for all unit vectors U at every point of M, then $h(p)^0 = A_p^0$.

Proof. Let $\tilde{M}$ be the universal covering manifold of M, and π be the projection mapping from $\tilde{M}$ to M, and g be the metric tensor field of M. Defining the tensor field $\tilde{g}$ on $\tilde{M}$ by $\tilde{g} = \pi^* g$, $\tilde{M}$ becomes a complete simply connected Riemannian locally symmetric space, which is a Riemannian globally symmetric space [2]. Since $\tilde{M}$ is a complete Riemannian manifold whose Ricci tensor $\tilde{K}$ satisfies $\tilde{K}(\tilde{U}, \tilde{U}) \geqq \alpha$ for all unit vectors $\tilde{U}$ at every point of $\tilde{M}$, $\tilde{M}$ is compact ([4]p. 105). If we denote by $\tilde{h}(\tilde{p})$ the linear holonomy group of $\tilde{M}$ at $\tilde{p}$, then from Theorem 2 we have $\tilde{h}(\tilde{p}) = \tilde{A}_{\tilde{p}}^0$. On the other hand, if $\pi(\tilde{p}) = p$, $\tilde{h}(\tilde{p}) = h(p)^0$ [3] and $\tilde{A}_{\tilde{p}} = A_p$. Therefore we have $h(p)^0 = A_p^0$.

Corollary 1. *If M is a complete Riemannian locally symmetric space, and for some positive number α the Ricci curvature K satisfies $K(U, U) \geqq \alpha$ for all unit vectors U at every point of M, then $h(p)^0 = A_p^0$.*

Corollary 2. *If a compact Riemannian globally symmetric space M has non zero sectional curvature, then $h(p)^0 = A_p^0$.*

Proof. Since the sectional curvature of a compact Riemannian globally symmetric space is non negative [1], now it is positive, and hence the Ricci curvature is also positive. Since M is compact, there exists a positive number α such that $K(U, U) \geqq \alpha$ for all unit vectors U at every point of M.

References

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