60. On Almost Periodic Transformations on Metric Space over Topological Semifield

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A. Edrei, P. Erdös, W. H. Gottschalk, G. A. Hedlund, and A. H. Stone have obtained interesting results on transformations on topological spaces (for references, see [5]). In this note, we shall consider some results in a metric space over a topological semifield (for related concepts, see [1] and [2]). Let X be a metric space over a topological semifield R. We denote the metric by ρ. Let f be a continuous mapping on X, i.e. $f(X)\subset X$.

We first repeat some definitions needed.

The mapping f is said to be *strongly almost periodic* if for a given neighborhood U there is a positive integer k such that every k consecutive positive integers contains an n satisfying $\rho(x, f^n(x))\in U$ for all $x\in X$.

In the definition of strongly almost periodicity, the positive integer k is independently taken for each point x of X. If k depends on each point x, we need a new definition.

A point x of X is said to be *almost periodic* under f (by W. H. Gottschalk [4]) if for a given neighborhood U, there is a positive integer k such that every k consecutive positive integers contains n satisfying $\rho(x, f^n(x))\in U$. If each point x is almost periodic under f, the mapping f is said to be *pointwise almost periodic*. For $x\in X$, the set $\bigcup_{n=-\infty}^{\infty} f^n(x)$ is called the *orbit* of x under f and the set $\bigcup_{n=0}^{\infty} f^n(x)$ is called the *semi-orbit* of x under f.

Under these concepts, we shall prove the following theorem which is formulated by P. Erdös and A. H. Stone [3].

Theorem. Let X be a totally bounded metric space over a topological semifield, and f a homeomorphism of X. If the set of all negative powers is equiuniformly continuous, then f is strongly almost periodic.

The proof is quite similar with that of Theorem III of P. Erdös and A. H. Stone [3].

To prove Theorem, we take a neighborhood U of 0 in R. Then there is a neighborhood W such that $W+W\subset U$. For W, there is a neighborhood V of 0 such that $\rho(f^{-m}(x), f^{-m}(y))\in W$ holds for x, y of $f^m(x)$ for which $\rho(x, y)\in V$. Here we can take V and W as saturated neighborhoods and $V\subset W$.

Since X is totally bounded, there is a finite partition $X_i(i = 1, 2, \cdots, r)$ of X such that each X_i is set of diameter less than W. For each m, we correspond a matrix of type $r \times r$: $A_m = (a_{ij}(m))$, where

$$a_{ij}(m) = \begin{cases} 1, & \text{if } f^m(X_i) \cap X_j \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, we have only finite distinct matrices. Let us suppose these matrices $A_1, A_2, \cdots, A_k$.

Let m be a positive integer, then there is a positive integer p such that $A_{m+k} = A_p$ and $p \leq k$. If n is taken as $m + k - p$, then we have $m \leq n < m + k$. Next we shall show that $\rho(x, f^n(x)) \in U$ for all $x \in X$.

For x of X, there are X_i, X_j for which $x \in X_i$, $f^{m+k}(x) \in X_j$. Hence $a_{ij}(m + k) = 1$, and so we have $a_{ij}(p) = 1$. This shows $f^p(X_i) \cap X_j \neq 0$. If $y \in f^p(X_i) \cap X_j$, then we have

$$\begin{aligned} \rho(x, f^n(x)) &\ll \rho(x, f^{-p}(y)) + \rho(f^{-p}(y), f^n(x)) \\ &= \rho(x, f^{-p}(y)) + \rho(f^{-p}(y), f^{m+k-p}(x)) \\ &\ll \delta(X_i) + \delta(f^{-p}(X_j)) \in V + W \subset W + W \subset U, \end{aligned}$$

where $\delta(A)$ is the diameter of A. Therefore f is strongly almost periodic.

Next we shall consider a fundamental theorem on orbits.

Theorem. If X is a metric space over a topological semifield R, and x of X is almost periodic under f, then the closure Y of the orbit of x under f is minimal under f.

Remark. Y of X is said to be minimal under f if $f(Y) = Y$ and Y does not contain a proper closed and invariant subset under f.

To prove it, we suppose that Y contains a proper closed and invariant subset Z. Then Z does not contain x, and Z is closed, hence there is a neighborhood $\Omega(x, U)$ of x which does not meet Z, and further there is a saturated neighborhood W of 0 such that $W + W \subset U$. For W, we can find a positive integer k such that every k consecutive positive integers contains n for which $\rho(x, f^n(x)) \in W$.

Further, for an element z of Z we can find a neighborhood V such that $\rho(z, y) \in V$ implies $\rho(f^i(z), f^i(y)) \in W$ for $i = 1, 2, \cdots, k$. For V, there is a positive integer p such that $\rho(z, f^p(x)) \in V$, from $z \in Y$. Therefore we have

$$\rho(f^i(z), f^{p+i}(x)) \in W.$$

On the other hand, we can find a positive integer q such that $1 \leq q \leq k$ and $\rho(x, f^{p+q}(x)) \in W$. Hence we have

$$\rho(x, f^q(x)) \ll \rho(x, f^{p+q}(x)) + \rho(f^{p+q}(x), f^q(z)) \in W + W.$$

This shows $f^q(z) \in \Omega(x, W + W) \subset \Omega(x, U)$. Z is invariant, so we have $f^q(z) \subset Z$.

We have a similar theorem on semi-orbits. To do so, we must replace *minimal* into *semi-minimal*. The exact definition of the semi-minimal set is: a subset Y of X is said to be *semi-minimal* under f, if the closure of semi-orbit of each y of Y is always Y.

Theorem. If X is a metric space over a topological semifield, and x of X is almost periodic under f, the closure Y of the semi-orbit of x under f is semi-minimal.

We have not any difficulty in the proof. Suppose that Y is not semi-minimal, then there is a point y of Y such that the closure of the semi-orbit of y is not Y. Further the closure Z of the semi-orbit of y does not contain x, i.e. x Z. Hence, we can find a neighborhood $\Omega(x, U)$ of x which does not meet Z. Next we take a saturated neighborhood W such that $W+W \subset U$. For W, we can find a positive integer k such that every set of k consecutive positive integers contains n for which $\rho(x, f^n(x)) \in W$. Since the mapping f is continuous, we can take a neighborhood V of 0 in R such that $\rho(x, x') \in V$ implies $\rho(f^i(x), f^i(x')) \in W$ for $i=1, 2, \cdots, k$, where $x' \in X$.

Y is the closure of the semi-orbit of x, so we take a n integer p such that $\rho(y, f^p(x)) \in V$. For p, we can find an integer q such that $1 \leq q \leq k$ and $\rho(x, f^{p+q}(x)) \in W$. Therefore we have

$$\rho(x, f^q(y)) \ll \rho(x, f^{p+q}(x)) + \rho(f^{p+q}(x), f^q(y)) \in W+W.$$

Hence we have $f^q(y) \in \Omega(x, U)$, which is impossible.

Consequently, we have the following theorem from the above two results.

Theorem. If X is a metric space over a topological semifield, and f is pointwise almost periodic, then f gives an orbit (or a semi-orbit) closure decomposition.

References

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