85. *Singular Cut-off Process and Lorentz Covariance*

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§ 1. **Introduction.** Singular cut-off process means to construct a sort of convolution (by A-integral) with A-integrable function or limit of them, namely it is to construct the infinite sum (of usual field functions) which play the same role as the field function suffering the usual cut-off [4] p 822. At the first step let's give the exact definition of it. Let $\psi(x)$ denote the quantized field function with the form

$$\psi(x)=(1/(2\pi)^{3/2})\Big\{\int(a^+(\vec{k})/\sqrt{2k_0})\cdot\exp i(\vec{k}\cdot\vec{x}-k_0t)d\vec{k} + \int(a(\vec{k})/\sqrt{2k_0})\cdot\exp(-i)(\vec{k}\cdot\vec{x}-k_0t)d\vec{k}\Big\} \cdots (1),$$

$(k_0=\sqrt{k_1^2+k_2^2+k_3^2+m^2})$, $\rho(\vec{x})$ denote the A-integrable function defined on the nowhere dense perfect set which is equivalent to a smooth function (by the meaning of distribution), and $\{\rho_n(\vec{x})\}$ denote the some fixed sequence of the above A-integrable functions with the limit $\bar{\rho}(\vec{x})$ [3] p 137.

Suppose that $\psi_\rho(x)$ is given by the form (means the infinite sum)

$$\psi_\rho(x)=(1/(2\pi)^{3/2})\Big\{\int(a^+(\vec{k})/\sqrt{2k_0})\cdot(A)\int\rho(\vec{x}')\exp i\{\vec{k}(\vec{x}-\vec{x}')-k_0t\}d\vec{x}'d\vec{k} + \int(a(\vec{k})/\sqrt{2k_0})\cdot(A)\int\rho(\vec{x}')\exp(-i)\{\vec{k}(\vec{x}-\vec{x}')-k_0t\}d\vec{x}'d\vec{k}\Big\} \cdots (2).$$

Definition 1. The operation constructing $\psi_\rho(x)$, (or $\lim_{n\to\infty}\psi_{\rho_n}(x)$) from $\psi(x)$ is called singular cut-off process.

The representation of cut-off (by using mollifier) and the above representations (1), (2) have a sort of ambiguities deduced from the lack of the exact definition of integral. Here, we use the Λ inhomogeneous Lorentz covariance defined in § 4 Def. 3 as Lorentz covariance. The judgement whether this Lorentz covariance is satisfied or not for three dimensional cut-off also depends on the interpretation of the problem related to the exact definition of δ function and being alike to "arrow's" paradox by Zenon. This judgement is related to the contradiction of the interpretation of rigid body in relativity theory [9] p 176. Though Λ inhomogeneous Lorentz covariance is the more weak condition than the usual Lorentz covariance, the cut-off (three dimensional case) by using smooth function's mollifier exerts the negative influence even on this. This negative influence is based on the lack of Haar measure in three dimensional manifold which is invariant for inhomogeneous Lorentz transform. Since the carrier of

$\bar{\rho}(\vec{x})=\{\rho_n(\vec{x})\}$ can be considered as the set of countable infinite points, it is deduced from the lack of the exact definition of cut-off (related to integral) that this negative influence is eliminated by using $\bar{\rho}(\vec{x})$ by the most original interpretation of δ functions.

If change of the total measure of nowhere dense perfect set by Lorentz transform is neglected from the regard as the infinite sum of descrete points, this Lorentz covariance is satisfied even for $\psi_\rho(x)$. Our purpose of this article is to give a sort of counter example for the negative relation (by common sense) between this Lorentz covariance and cut-off by analyzing these concepts instead of giving the exact definition of integral needed for quantum field theory.

The materials of this article are arranged as follows: in § 2 we show the difference among three interpretations of δ function in [1] p 58 which is required for the interpretation of the carrier of singular mollifiers, § 3 contains the construction of an example of the singular sequencial mollifier $\bar{\rho}(\vec{x})=\{\rho_n(\vec{x})\}$ used for singular cut-off process (by using the construction of A-integral representation of distribution), and in § 4 we show the three dimensional singular cut-off process satisfying the Λ inhomogeneous Lorentz covariance.

§ 2. δ function. P. A. M. Dirac gives the following three interpretations of δ function.

(i) δ function is a quantity depending on a parameter x satisfying the conditions $\int_{-\infty}^{+\infty}\delta(x)dx=1$, $\delta(x)=0$ for $x\neq 0$ [1] p 58.

(ii) δ function is one which has the property $\langle\delta(x),\varphi(x)\rangle=\int_{-\infty}^{+\infty}\delta(x)\varphi(x)dx=\varphi(0)$, for any continuous (testing) function $\varphi(x)$ [1] p 59 [2] I, p 25.

(iii) δ function is the sequence (or the resemble one) of regular functions [1] p 58 (for example $\rho_{1/n}(x)$ by L. Schwartz [2] I, p 22). Here (ii) is the interpretation by the result deduced from (i), and (iii) is the concrete construction of type (i)'s δ function.

Since the meaning of $\int_{-\infty}^{+\infty}$ and $\delta(0)$ are not necessarily obvious, (i) cannot be used as the exact definition but satisfies the Λ inhomogeneous Lorentz covariance. L. Schwartz has defined the distribution (which containes $\delta(x)$) by giving the rigourous definition of testing function's space which is alike to the space in (ii), [2] I, p 70. But, since his $\delta(x)$ is represented by using A-integrable function very exactly, it is shown that the definition of δ function by L. Schwartz suffers the effect by the part $x\neq 0$ which can be also seen from the definition of the distribution's carrier by using testing function, and this $\delta(x)$ gives the negative effect to Λ inhomogeneous Lorentz covariance (see § 4). As an example of definition (iii), P. A. M. Dirac uses the

Féjer kernel considered as the fixed sequence [1] p 95. The set of the sequences can be also considered as the δ function [8] p 329. But even if any one of these two interpretations of (iii) is used, Λ inhomogeneous Lorentz covariance cannot be necessarily satisfied. However many physicians and mathematicians have a sort of common sense (not necessarily rigourous) such that the above three definitions are equivalent.

§ 3. **The singular sequencial mollifier.** E_1 is same as the nowhere dense perfect set E defined in [4] p 823. Let E_n denote the $1/n$ similar restriction of E_1. The set F_1 is constructed by the arrangement of the translation of E_1 into all intervals $[2\nu, 2\nu+1]$($\nu=0, \pm 1, \pm 2, \cdots$). Next, the set F_2 is constructed by the arrangement of the translation of E_{8^1} in the middle of all the open connected components of F_1 whose length is larger than $3/2^1$. Iterating the same constructing processes (countable infinite times) the set F_∞ is constructed.

For a decreasing positive number's sequence $\{\varepsilon_l\}$ tending to zero, the following subsequence $\{F_{n'_l}\}$ (of the sequence $\{F_n\}$) can be chosen. Namely $F_{n'_{l+1}}-F_{n'_l}$ is the set which is the join of the disjointed nowhere dense perfect sets $E_{p,k,\nu}$ with the following properties; (1) the distance between two neighbouring sets of the family $\{E_{p,k,\nu}; k=1, 2, \cdots K_p, n'_l<p\leqq n'_{l+1}, \nu=0, \pm 1, \cdots\}$ is smaller than $\varepsilon_l>0$ [5] p 76 Lemma 2, (2) the length $l(E_{p,k,\nu})$ of each $E_{p,k,\nu}$ contained in the above family satisfies the inequality $8^{-n'_l}>l(E_{p,k,\nu})\geqq 8^{-n'_{l+1}}$.

An A-integral representation of distribution is the sum of the elements of the sequence $\{f_l(x)\}$. Here, $f_l(x)$ is the function with the range of the values $\{\pm n_l\}$ whose carrier σ_l is the subset of $F_{n'_{l+1}}-F_{n'_l}$, and this function has the same sign in $E_{p,k,\nu}$ (for the same p, k, ν) [5] p 77.

The following $\{f_l^{(2)}(x)\}$ representing the same distribution can be constructed from this sequence $\{f_l(x)\}$: the values of $f_l^{(2)}(x)$ are $\pm 2n_l$, and the carrier $\sigma_l^{(2)}$ of $f_l^{(2)}(x)$ is the set $\bigcup_{n'_l<p\leqq n'_{l+1}}\bigcup_{1\leqq k\leqq K_p}\bigcup_{\nu=0,\pm 1,\cdots}E'_{p,k,\nu}$ where

$$E'_{p,k,\nu}=\left\{y; y\in E_{p,k,\nu}\cap\sigma_l, \int_{(-\infty,y)\cap E_{p,k,\nu}\cap\sigma_l}dx\leqq\int_{E_{p,k,\nu}\cap\sigma_l}dx/2\right\} \qquad [5] \text{ p } 77.$$

By using the same method etc., we can construct the sequence of the sequences $[\{f_l^{(m)}(x)\}]$ with the property such that $f_l^{(m)}(x)$ (for $m>n(L)$, for arbitrary fixed l depending on L) have the same sign in the fixed set $E_{p,k,\nu}$ $(1\leqq p\leqq L, 1\leqq k\leqq K_p, \nu=0, \pm 1, \pm 2, \cdots)$.

Here $n(L)$ is the monotone increasing integer valued function of L depending on the order q (of distribution) and $\{\varepsilon_l\}$, [5] p 77 [3] p 137. For example, $n(L)=2([1/\varepsilon_L]+2)^{2q}$, where $[1/\varepsilon_L]$ is the maximum positive integer smaller than (or equal to) $1/\varepsilon_L$.

From the above argument we assert the following

Theorem 1. *Suppose that* $\{f^{(m)}(x)\}=\{\sum_{l=}^{\infty} f_{1l}^{(m)}(x)\}$ *is the A-integral representation's sequence of a fixed distribution. The sequence* $\{f^{(m)}(x)\}$ *with the following properties can be constructed;*

(1) *for* $m>n$ $\{\text{Carrier } f^{(m)}(x)\}\subset\{\text{Carrier } f^{(n)}(x)\}$,

(2) $\bigcap_n \{\text{Carrier } f^{(n)}(x)\}$ *is the set of countable infinite points, and* $\lim_{m\to\infty}\sum_{l=1}^{\infty} f_l^{(m)}(x)$ *can be considered as the countable infinite sum of δ-like singularities (of the type* (i)).

We can also understand that $\lim_{m\to\infty}\sum_{l=1}^{\infty} f_l^{(m)}(x)$ has the sequence $\{\sum_{l=1}^{\infty} f_l^{(m)}(x)\}$ showing its construction. Namely the singular cut-off process by using $\lim_{m\to\infty}\sum_{l=1}^{\infty} f_l^{(m)}(x)$ is one by using the sequence $\{\sum_{l=1}^{\infty} f_l^{(m)}(x)\}$. On the other hand, this singular cut-off can be also defined as the conditional convergent series of the countable infinite sum of type (i)'s δ-like singularities in $\lim_{m\to\infty}\sum_{l=1}^{\infty} f_l^{(m)}(x)$.

§ 4. Λ inhomogeneous Lorentz covariance. Here we treat the following Hyperbolic differential equation (3) as the extension of the equation for fixed point source model with
$\psi(x)\equiv g\delta(\vec{x})$ [7] p 2; $\quad(\vec{\nabla}^2-\partial^2/\partial t^2-\mu^2)\Phi(x)=\psi(x)\cdots(3)$.
Hereafter, let $\vec{x}$ denote the three dimensional vector, and $x=(\vec{x}, t)$.

Suppose that $U(a, \Lambda)$ are the unitary operator defined in the state vector's space [10] p 22 (or Von Neumann's direct product space) and deduced from the inhomogeneous Lorentz transform (a, Λ).

Definition 2. If there exists $U(a, \Lambda)$ such that the relation $(\Phi, \varphi(x)\Psi)=(U\Phi, U\varphi(x)\Psi)=(U(a, \Lambda)\Phi, \varphi(a+\Lambda x)U(a, \Lambda)\Psi)$ holds good for any state vectors Φ and Ψ, then we say that $\varphi(x)$ satisfies the Lorentz covariance [6] p 249 [10] p 99.

Definition 3. If $\varphi(x)*\rho(x)$ satisfies the Lorentz covariance by the replacing $(\rho(x)\to\rho(\Lambda x))$, we say that $\varphi(x)*\rho(x)$ satisfies the Λ inhomogeneous Lorentz covariance.

This Definition 3 shows the Lorentz covariance for the class $\{\varphi(x)*\rho(\Lambda x); \Lambda\in L\}$, where L is the homogeneous Lorentz group. This is a sort of specialized form of Y. Kato's Lorentz covariance for $(\mathfrak{S}')$.

Lemma 1. *If $\Psi(x)$ satisfies Lorentz covariance, then $\Psi(x)*\rho(x)$ also satisfies Λ inhomogeneous Lorentz covariance for usual function $\rho(x)$.*

Proof. Since

$$U^{-1}(a, \Lambda)\int\psi(x-x')\rho(x')dx'\,U(a, \Lambda)=\int\psi(\Lambda x+a-\Lambda x')\rho(x')dx'$$

$=\int\psi(\Lambda x+a-\Lambda x')\rho(x')d(\Lambda x')$ holds valid, this Lemma is concluded.

Furthermore we show the following Lemma without proof.

Lemma 2. *If and only if $\psi(x)$ satisfies the usual Lorentz covariance, the quantized solution of* (3) *also satisfies the usual*

Lorentz covariance.

(a) Since the unit length of t is varied everywhere, it can be easily seen that the Lorentz covariance is not satisfied for $\int \psi(x-\vec{x}')\rho(\vec{x}')d\vec{x}'$, where $\rho(\vec{x}')$ is the usual function. Namely $U^{-1}(a, \Lambda)\int \psi(\vec{x}-\vec{x}', t)\rho(\vec{x}')d\vec{x}'U(a, \Lambda)=\int \psi(\Lambda x-\overrightarrow{\Lambda(\vec{x}', 0)}+a)\rho(\vec{x}')d\vec{x}'$. By the replacing $(\rho(\vec{x}')\to\rho(\overrightarrow{\Lambda(\vec{x}', 0)}))$, this result becomes to

$$\int \psi(\Lambda x+a-\overrightarrow{\Lambda(\vec{x}', 0)})\rho(\overrightarrow{\Lambda(\vec{x}', 0)})\left(\frac{\partial(\overrightarrow{\Lambda(\vec{x}', 0)})}{\partial \vec{x}}\right)^{-1} d(\overrightarrow{\Lambda(\vec{x}', 0)}),$$

where $d(\overrightarrow{\Lambda(\vec{x}', 0)})$ means the three dimensional integral.

(b) If $\rho(\vec{x}')$ is the finite sum of δ-like singularities of type (i) (defined in § 2), $U^{-1}(a, \Lambda)\sum_i \psi(\vec{x}-\vec{x}_i', t)U(a, \Lambda)=\sum_i \psi(\Lambda x+a-\overrightarrow{\Lambda(\vec{x}_i', 0)})\to \sum_i \psi(\Lambda x+a-\vec{x}_i')$ for $(\rho(x)\to\rho(\Lambda x))$. Then Λ inhomogeneous Lorentz covariance is satisfied. This result is deduced from the assumption such that type (i)'s δ function is translated by Lorentz transform. This assumption is deduced from the property such that the measure of the (type (i)'s) $\delta(\vec{x})$'s carrier is zero truely. The singular cut-off process (2) by $\bar{\rho}=\{\rho_n\}$ in § 3 rather has the property of this finite sum in itself.

(c) If $\rho(\vec{x}')$ is the δ function of type (ii) (A-integral representation), or the δ function of type (iii) (defined in § 2), the relation in (a) is satisfied from the same argument (in (a)).

Hence we assert the following

Theorem 2. *Suppose that $\bar{\rho}(\vec{x}')$ is the singular sequencial mollifier (for example, see* § 3) *constructed by using the δ function of type* (i) (*see* § 2). *Then, the Λ inhomogeneous Lorentz covariance is satisfied for singular cut-off by using $\bar{\rho}(\vec{x}')$.*

Proof. Since mes {Carrier $(\bar{\rho}(\vec{x}'))$} is zero, this measure does not vary through the Lorentz transform. After the Lorentz transform, the infinite sum of the δ functions of type (i) defined on the descrete points set is also considered as the infinite sum of the translated δ functions with the same properties as one before the Lorentz transform from the same argument as the above (b). Hence Lorentz covariance by the meaning of (b) is satisfied.

In Theorem 2 the effective condition to satisfy this Lorentz covariance (and causality) is that carrier $(\bar{\rho}(\vec{x}'))$ is the countable infinite points. Furthermore, it is effective that all points in this carrier lies in boundary. If we neglect the variance of the infinitesimal measure related to each separate point, this Lorentz covariance can be also satisfied by using A-integrable function itself (defined on the countable sum of nowhere dense perfect set) as mollifier. This argument corresponds to the use of Dini's derivate for generalized causal set [5] p 74.

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