81. On a Certain Functional-Differential Equation

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1. Let $\mathfrak{M}$ be a family of functions continuous in I: $0 \leqq t < \infty$ in the n-dimensional vector space. Then, we define an operator T satisfying the following conditions:

(i) for any x in $\mathfrak{M}$, Tx is also contained in $\mathfrak{M}$;

(ii) for any sequence $\{x_m\}$ ($x_m \in \mathfrak{M}$) uniformly convergent in I, $\{Tx_m\}$ is also uniformly convergent in I;[1]

(iii) for any scalar functions u and v continuous in I, if $u \leqq v$ is satisfied for $0 \leqq t < s$, where s is an arbitrary constant, then the inequality $Tu \leqq Tv$ remains valid for $t = s$.

Then, let us consider a functional-differential equation such that

(1) $$x' = f(t, x, Tx), \quad x(0) = x_0, \quad t \in I.$$

If we choose the operator and the function f suitably, the equation (1) yields various types of equations, for example, differential equations, integro-differential equations, difference-differential equations, and so on.

In the sequel, the existence of continuous solutions of (1) in I is supposed to be established. However, we need not assume the uniqueness of solutions, so far as we are concerned with the boundedness and stability problems.[2]

2. We first introduce a V-function as follows. Let $V(t, x)$ be a function of t and x satisfying the following conditions:

(i) $V(t, x)$ is continuous and non-negative in I and $|x| < \infty$;

(ii) $V(t, x)$ satisfies the Lipschitz condition such that

$$|V(t, x) - V(t, y)| \leqq k(t)\,|x - y|,$$

where $k(t)$ is continuous in I;

(iii) $\lim_{|x| \to \infty} V(t, x) = \infty$ uniformly in $t \in I$.

In order to derive some results on the boundedness, it is usefull to introduce two quantities $\mathfrak{d}V(t, x, y)$ and $DV(t, x)$ by setting

$$\mathfrak{d}V(t, x, y) = \overline{\lim_{h \to 0}} \frac{1}{h}(V(t+h, x+hf(t, x, y)) - V(t, x)),$$

$$DV(t, z(t)) = \overline{\lim_{h \to 0}} \frac{1}{h}(V(t+h, z(t+h)) - V(t, z(t))),$$

1) This means that the operator T is continuous.

2) The author's paper, in which some theorems on the existence and uniqueness of continuous solutions has been discussed, will shortly appear.

where x, y are arbitrary points, and $z(t)$ is an arbitrary function in $\mathfrak{M}$. Then, it is not difficult to prove that the equality

$$(2) \qquad \mathfrak{d} V(t, x(t), (Tx)(t)) = DV(t, x(t))$$

remains valid in I for any solution $x(t)$ of (1).

Now, we define a function $\omega(t, x, y)$ such that it is continuous and non-negative for I, $0 \leqq x < \infty$, $-\infty < y < \infty$. Furthermore, it is supposed that $\omega(t, x, y)$ is monotone increasing with respect to y for any fixed t and x. With this choice of the function $\omega(t, x, y)$, for any non-negative constant $\varepsilon \geqq 0$, we consider a functional-differential equation such that

$$(3) \qquad r' = \omega(t, r, Tr) + \varepsilon, \quad r(0) = r_0 + \varepsilon, \quad t \in I,$$

for which the existence of the maximal solution $r_\varepsilon(t)$ of (3) continuous in I is supposed to be established.

Theorem 1. *If the inequality*

$$(4) \qquad \mathfrak{d} V(t, x, Tx) \leqq \omega(t, V(t, x), (TV)(t, x))$$

is fulfilled for any $t \in I$ and x, the relation

$$(5) \qquad V(t, x) \leqq r_0(t)$$

remains valid in I for any continuous solution x of (1), *provided that $V(0, x_0) \leqq r_0$ is satisfied.*

Proof. From the continuity of $x(t)$, $r_\varepsilon(t)$, and V, it turns out that there exists an interval $0 \leqq t < t_1$, in which the inequality $V(t, x(t)) \leqq r_\varepsilon(t)$ remains valid for any solution $x(t)$ of (1).

Then, if we denote by t_0 the supremum of t_1, and if t_0 is finite, from the continuity of V, x, r_ε, and from (2), (4), it follows that

$$\begin{aligned}
& V(t_0, x(t_0)) = r_\varepsilon(t_0), \\
& \omega(t_0, V(t_0, x(t_0)), (TV)(t_0, x(t_0))) + \varepsilon \\
\leqq\; & \omega(t_0, r(t_0), (Tr_\varepsilon)(t_0)) + \varepsilon \\
=\; & r'_\varepsilon(t_0) \\
=\; & \lim_{t \to t_0} \frac{r_\varepsilon(t) - r_\varepsilon(t_0)}{t - t_0} \\
\leqq\; & \overline{\lim_{t \to t_0}} \frac{V(t, x(t)) - V(t_0, x(t_0))}{t - t_0} \\
=\; & DV(t_0, x(t_0)) \\
=\; & \mathfrak{d} V(t_0, x(t_0), (Tx)(t_0)) \\
\leqq\; & \omega(t_0, V(t_0, x(t_0)), (TV)(t_0, x(t_0))),
\end{aligned}$$

which is a contradiction, since $\varepsilon > 0$. Hence, the inequality

$$V(t, x) \leqq r_\varepsilon(t)$$

is fulfilled in I. Since $r_\varepsilon(t)$ is monotone decreasing as $\varepsilon \to +0$, it uniformly converges to the maximal solution $r_0(t)$ of (3) corresponding to $\varepsilon = 0$. Thus, we have the inequality (5) in I.

By means of Theorem 1, we obtain the following

Corollary. *If the maximal solution of* (3) *is bounded, the solu-*

tions of (1) *are bounded.*

Proof. On the contrary, if we suppose that a solution $x(t)$ of (1) is not bounded, there exists a sequence $\{t_k\}$, $t_k\to\infty$ such that $|x(t_k)|\to\infty$. Then, from the inequality (5) and the definition (iii) of V, it follows that

$$r_0(t_k)\geqq V(t_k, x(t_k))\to\infty,$$

which contradicts the boundedness of $r_0(t)$.

If we consider a particular case such that $V=|x|$, the inequality (4) is reduced to

$$|f(t, x, Tx)|\leqq\omega(t, |x|, |Tx|).$$

Furthermore, suppose that $\omega(t, x, y)$ is of the form such that

$$\omega(t, x, y)=k(t)(M(|x|)+M(|y|)),$$

where $k(t)$ is continuous in I and $M(r)$ is piecewise continuous, positive, non-decreasing for $0<r<\infty$, and $M(0)=0$. Then, we have the following

Theorem 2. *Suppose that the inequality*

$$|f(t, x, y)|\leqq k(t)(M(|x|)+M(|y|))$$

is satisfied and $k(t)$ *is integrable over* I, *but the integral*

$$\int^{\infty}\frac{d\rho}{M(\rho)}$$

is divergent. Then, if T *is the bounded operator, any solution of* (1) *is bouuded.*

Proof. Since T is bounded, there exists a constant $\alpha>0$ such that $|Tx|\leqq\alpha|x|$ for any $x\in\mathfrak{M}$. Then, if we consider an equation such that

(6) $$r'=k(t)(M(r)+M(|Tr|)),$$

for any solutions of r of (6) we have the inequalities such that

(7) $$\begin{aligned} r'&\leqq k(t)(M(r)+M(\alpha r))\\ &\leqq 2k(t)M(r) \qquad (0<\alpha\leqq 1),\\ &\leqq 2k(t)M(\alpha r) \qquad (1\leqq\alpha<\infty).\end{aligned}$$

From (7), it follows along the solution r that

$$\int_{r_0}^{r}\frac{d\rho}{M(\rho)}\leqq 2\int_0^t k(s)ds$$

for $0<\alpha\leqq 1$, and

$$\int_{\alpha r_0}^{\alpha r}\frac{d\rho}{M(\rho)}\leqq 2\int_0^t k(s)ds$$

for $1\leqq\alpha<\infty$. Hence, it follows that r must be bounded in I.