98. On Axiom Systems of Propositional Calculi. II

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The purpose of this paper is to present the proofs of equivalences of some postulate sets for two valued propositional calculus. In this paper we use the well known technique in the propositional calculus of J. Lukasiewicz (For the details, see J. Lukasiewicz [1]). We shall prove that if we adopt the rule of procedure ordinarily used in the propositional calculus, i.e. the rule of substitution and the rule of detachment adjusted to the implication, then any thesis is a consequence of the following three axioms:

1 $CCpqCCqrCpr$,

2 $CpCNpq$,

3 $CCNpqCCqpp$.

J. Lukasiewicz derived from the axiom system, given by him in 1924, thesis 3 of our axiom system. We shall now proceed to prove thesis 2 of his axiom system in 1924, i.e. $CCNppp$, from the adopted postulate sets.

Then the proofs may be recognized that our axiom system consisting of theses 1, 2, and 3 would suffice to build the propositional calculus, in doing so we shall confine ourselves to the three axioms and the two rules of inference given above.

The proofs will be carried out in the complete form without any gaps. Every derived thesis will have its number and will be preceded by a proof line.

Using the rules of procedure we can prove from the adopted axioms:

1 $CCpqCCqrCpr$,

2 $CpCNpq$,

3 $CCNpqCCqpp$,

the following theses:

$1\, p/Cpq,\ q/CCqrCpr,\ r/s * C1—4,$

4 $CCCCqrCprsCCpqs$.

$4\, s/CCCprsCCqrs * C1\, p/Cqr,\ q/Cpr,\ r/s—5,$

5 $CCpqCCCprsCCqrs$.

$4\, q/Cqr,\ r/Csr,\ s/CCsqCpCsr * C4\, p/s,\ s/CpCsr—6,$

6 $CCpCqrCCsqCpCsr$.

$6\, p/Cpq,\ q/CCprs,\ r/CCqrs,\ s/t * C5—7,$

7 $CCtCCprsCCpqCtCCqrs$.

$1\ q/CNpq * C2$—8,

8 $CCCNpqrCpr$.

$8\ r/CCqpp * C3$—9,

9 $CpCCqpp$.

$6\ p/q,\ q/CNpq,\ r/q,\ s/p * C9\ p/q,\ q/Np$—$C2$—10,

10 $CqCpq$.

$1 p/q,\ q/Cpq * C10$—11,

11 $CCCpqrCqr$.

$11\ p/Nq,\ q/p,\ r/CCpqq * C3\ p/q,\ q/p$—12,

12 $CpCCpqq$.

$6\ p/q,\ q/Cqr,\ s/p * C12\ p/q,\ q/r$—13,

13 $CCpCqrCqCpr$.

$1\ p/CpCqr,\ q/CqCpr,\ r/s * C13$—14,

14 $CCCqCprsCCpCqrs$.

$14\ q/Np,\ r/q,\ s/CCCpqpp * C3\ q/Cpq$—$C2$—15,

15 $CCCpqpp$.

$13\ p/Cpq,\ q/CCprs,\ r/CCqrs * C5$—16,

16 $CCCprsCCpqCCqrs$.

$16\ p/Cpq,\ r/p,\ s/p,\ q/r * C15$—17,

17 $CCCpqrCCrpp$.

$17\ r/q * 18$,

18 $CCCpqqCCqpp$.

$1\ p/CCpqr,\ q/CCrpp,\ r/s * C17$—19,

19 $CCCCrppsCCCpqrs$.

$19\ s/CCprr * C18\ p/r,\ q/p$—20,

20 $CCCpqrCCprr$.

$7\ t/CCpqr,\ s/r,\ q/s * C20$—21,

21 $CCpsCCCpqrCCsrr$.

$13\ p/Cps,\ q/CCpqr,\ r/CCsrr * C21$—22,

22 $CCCpqrCCpsCCsrr$.

$22\ q/r,\ r/CqCpr * C10\ q/Cpr,\ p/q$—23,

23 $CCpsCCsCqCprCqCpr$.

$13\ p/Cps,\ q/CsCqCpr,\ r/CqCpr, * C23$—24,

24 $CCsCqCprCCpsCqCpr$.

$13\ p/Cpq,\ q/Cqr,\ r/Cpr * C1$—25,

25 $CCqrCCpqCpr$.

$24\ s/Cqr,\ q/Cpq * C25$—26,

26 $CCpCqrCCpqCpr$.

Having proved theses 4–26 we are now in a position to deduce thesis 27:

$26\ r/p * C10 q/p,\ p/q$—27,

27 $CCpqCpp$.

Then we prove the next theorem:

$$27\ q/Cqp * C10q/p,\ p/q—28,$$

28 Cpp.

Thesis 28 is called the *law of identity*. By this theorem and thesis 3 we can derive thesis 29 which occurs as an axiom in the axiom system of the propositional calculus, given by J. Lukasiewicz in 1924.

$$3\ q/Np * C28\ p/Np—29,$$

29 $CCNppp$.

Hence the abbreviation in the proof of thesis 29 can be written down without using thesis 28. We shall again proceed to prove thesis 29 by the following proof lines:

$$14\ q/Np,\ r/p,\ s/CCCpppp * C3\ q/Cpp—C2\ q/p—30,$$

30 $CCCpppp$.

We can prove thesis 29 by theses 1, 3, and 30:

$$1\ p/CNpp,\ q/CCppp,\ r/p * C3\ q/p—C30—29,$$

29 $CCNppp$.

In the future papers we shall take up another postulate sets for propositional calculus, and consider how to for mulate these postulate sets to algebraic systems.

Reference

[1] J. Lukasiewicz: Elementy logiki matematycznei (Elements of Mathematical Logic). Warszawa, PWN, 2nd ed. (1958).