124. On Axiom Systems of Propositional Calculi. III

By Yoshinari ARAI

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In this note, we shall concern with Lukasiewicz (L_3) axioms (see Y. Imai and K. Iséki [1]). As mentioned in our previous papers, we only use the rules of substitution and detachment. The fundamental axioms are the following three theses:

1 $CpCqp$,

2 $CCpCqrCCpqCpr$,

3 $CCNpNqCqp$.

We shall first give a proof of $(L_3)\Rightarrow(L_1)$. From the (L_3)-system, we have the following theses:

1 $p/CCNqNpCpq$, q/Np $*C3$ p/q, q/p—4,

4 $CNpCCNqNpCpq$.

2 p/Np, $q/CNqNp$, r/Cpq $*C4$—$C1$ p/Np, q/Nq—5,

5 $CNpCpq$.

2 p/Nq, r/p $*C5$ p/q, q/p—6,

6 $CCNqqCNqp$.

1 $p/CCNqqCNqp$ $*C6$—7,

7 $CqCCNqqCNqp$.

2 p/q, $q/CNqq$, $r/CNqp$ $*C7$—$C1$ p/q, q/Nq—8,

8 $CqCNqp$.

8 q/p, p/q $*C9$,

9 $CpCNpq$.

1 $p/CCpCqrCCpqCpr$, q/Cqr $*C2$—10,

10 $CCqrCCpCqrCCpqCpr$.

2 p/Cqr, $q/CpCqr$, $r/CCpqCpr$ $*C10$—$C1$ p/Cqr, q/p—11,

11 $CCqrCCpqCpr$.

2 p/Cqr, q/Cpq, r/Cpr $*C11$—12,

12 $CCCqrCpqCCqrCpr$.

1 $p/CCCqrCpqCCqrCpr$, q/Cpq $*C12$—13,

13 $CCpqCCCqrCpqCCqrCpr$.

2 p/Cpq, $q/CCqrCpq$, $r/CCqrCpr$ $*C13$—$C1$ p/Cpq, q/Cqr—14,

14 $CCpqCCqrCpr$.

2 $p/CpCqr$, q/Cpq, r/Cpr $*C2$—15,

15 $CCCpCqrCpqCCpCqrCpr$.

2 r/p $*C1$—16,

16 $CCpqCpp$.

16 q/Cqp *C1—17,

17 Cpp.

1 p/Cpp, $q/CpCpq$ *C17—18,

18 $CCpCpqCpp$.

15 q/p, r/q *C18—19,

19 $CCpCpqCpq$.

11 q/Cpq, r/Cpr, p/q—20,

20 $CCCpqCprCCqCpqCqCpr$.

1 $p/CqCpq$, $q/CCpqCpr$ *C1 p/q, q/p—21,

21 $CCCpqCprCqCpq$.

2 $p/CCpqCpr$, $q/CqCpq$, $r/CqCpr$ *C20—C21—22,

22 $CCCpqCprCqCpr$.

11 $q/CCpqCpr$, $r/CqCpr$, $p/CpCqr$ *C22—C2—23,

23 $CCpCqrCqCpr$.

11 q/Cpq, $r/CCqrCpr$, $p/CNqNp$ *C14—C3 p/q, q/p—24,

24 $CCNqNpCCqrCpr$.

11 $q/CNqNp$, $r/CCqrCpr$, $p/CNqq$ *C24—C6 p/Np—25,

25 $CCNqqCCqrCpr$.

23 $p/CNqq$, q/Cqq, $r/CCNqqq$ *C25 r/q,
$p/CNqq$—C17 p/q—26,

26 $CCNqqCCNqqq$.

19 $p/CNpp$, q/p *C26 q/p—27,

27 $CCNppp$.

In the theses given above, theses 9, 14, and 27 are the axioms of (L_1)-system. Then it follows that (L_1)-system is equivalent to (L_3)-system.

Next we shall prove the implications: (L_3)$\Rightarrow$(R), (L_2), (H), (S_1), (F), and (S_2).

We prove further:

14 p/Np, q/Cpq *C5—28,

28 $CCCpqrCNpr$.

28 p/Np, q/p, r/p *C27—29,

29 $CNNpp$.

14 p/Cpq, $q/CCqrCpr$, r/s *C14—30,

30 $CCCCqrCprsCCpqs$.

30 q/Cqr, r/Csr, $s/CCsqCpCsr$ *C30 p/s, $s/CpCsr$—31,

31 $CCpCqrCCsqCpCsr$.

30 $s/CCCprsCCqrs$ *C14 p/Cqr, q/Cpr, r/s—32,

32 $CCpqCCCprsCCqrs$.

31 p/Cpq, $q/CCprs$, $r/CCqrs$, s/t *C32—33,

33 $CCtCCprsCCpqCtCCqrs$.

14 $q/CNpq$ *C9—34,

34 $CCCNpqrCpr$.

34 $r/CCCNpppCCqpp$ *$C32$ p/Np, r/p, s/p—35,

35 $CpCCCNpppCCqpp$.

35 $p/CCNppp$ *$C27$—$C27$ $p/CCNppp$—36,

36 $CCqCCNpppCCNppp$.

34 p/t, $q/CCNppp$, $r/CCNppp$ *$C36$ q/Nt—37,

37 $CtCCNppp$.

33 p/Np, r/p, s/p *$C37$—38,

38 $CCNpqCtCCqpp$.

14 $p/CNpq$, $q/CtCCqpp$ *$C38$—39,

39 $CCCtCCqpprCCNpqr$.

39 $t/NCCqpp$, $r/CCqpp$ *$C27$ $p/CCqpp$—40,

40 $CCNpqCCqpp$.

14 $p/CpCqr$, $q/CqCpr$, r/s *$C23$—41,

41 $CCCqCprsCCpCqrs$.

41 q/Np, r/q, $s/CCCpqpp$ *$C40$ q/Cpq—$C9$—42,

42 $CCCpqpp$.

23 p/Cpq, $q/CCprs$, $r/CCqrs$ *$C32$—43,

43 $CCCprsCCpqCCqrs$.

43 p/Cpq, r/p, s/p, q/r *$C42$—44,

44 $CCCpqrCCrpp$.

44 p/Np, q/p, r/p *$C27$—45,

45 $CCpNpNp$.

14 p/NNp, q/p, r/q *$C29$—46,

46 $CCpqCNNpq$.

14 p/Cpq, $q/CNNpq$ *$C46$—47,

47 $CCCNNpqrCCpqr$.

47 $r/CCqNpNp$ *$C40$ p/Np—48,

48 $CCpqCCqNpNp$.

31 p/Cpq, $q/CqNp$, r/Np *$C48$—49,

49 $CCsCqNpCCpqCsNp$.

49 s/q, q/Nq *$C9$ p/q, q/Np—50,

50 $CCpNqCqNp$.

Theses 14, 23, 29, 45, 50, and the axiom 1 of (L_3) are the axioms of (R)-system.

Moreover, from theses 1, 5, and 14, we have the following theses:

14 p/Np, q/Cpq *$C5$—51,

51 $CCCpqrCNpr$.

14 p/q, q/Cpq *$C1$ p/q, q/p—52,

52 $CCCpqrCqr$.

Hence the axioms 1 and 2 of (L_2)-system are derived from (L_3)-system. We now prove the third axiom of (L_2)-system.

31 $p/CNpq$, q/Cqp, r/p *$C40$—53,

53 $CCsCqpCCNpqCsp$.

53 s/Nq *$C5$ p/q, q/p—54,

54 $CCNpqCNqp$.

14 $p/CNpq$, $q/CNqp$ *$C54$—55,

55 $CCCNqprCCNpqr$.

55 $r/CCpqq$ *$C40$ p/q, q/p—56,

56 $CCNpqCCpqq$.

23 $p/CNpq$, q/Cpq, r/q *$C56$—57,

57 $CCpqCCNpqq$.

We complete the proof of $(L_3)\Rightarrow(H)$. Because theses 9, 11, 23, 57, and the axiom 1 of (L_3) are the (H)-system.

Our proof would be run:

14 $p/CCpqr$, $q/CCrpp$, r/s *$C44$—58,

58 $CCCCrppsCCCpqrs$.

58 $s/CCprr$ *$C44$ p/r, q/p, r/p—59,

59 $CCCpqrCCprr$.

33 $t/CCpqr$, s/r, q/s *$C59$—60,

60 $CCpsCCCpqrCCsrr$.

23 p/Cps, $q/CCpqr$, $r/CCsrr$ *$C60$—61,

61 $CCCpqrCCpsCCsrr$.

14 $p/CNpq$, $q/CCpqq$ *$C56$—62,

62 $CCCCpqqrCCNpqr$.

62 q/r, $r/CCpqCCqrr$ *$C61$ q/r, s/q—63,

63 $CCNprCCpqCCqrr$.

14 $p/CNpr$, $q/CCpqCCqrr$, r/s *$C63$—64,

64 $CCCCpqCCqrrsCCNprs$.

64 $s/CCqrCCpqr$ *$C23$ p/Cpq, q/Cqr—65,

65 $CCNprCCqrCCpqr$.

Theses 65 occurs as the axiom 3 of (L_2). Consequently we have completed the proof of $(L_3)\Rightarrow(L_2)$.

We may also give a proof of $(L_3)\Rightarrow(S_1)$. The axioms 1 and 3 of (S_1) are already proved by obtained theses 5 and 65. The second axiom of (S_1) will be derived from theses 1 and 23 by the following proof line:

1 $p/CpCrp$ *$C1$ q/r—66,

66 $CqCpCrp$.

23 p/q, q/p, r/Crp *$C66$—67,

67 $CpCqCrp$.

From the theses already shown, we have the following theses:

49 s/Nq *$C5$ p/q, q/Np—68,

68 $CCpqCNqNp$.

34 q/NNp, r/NNp *$C45$ p/Np—69,

69 $CpNNp$.

Theses 1, 2, 29, 68, and 69 form the axioms of (F)-system.

We prove an important thesis $CCpqCNqCpr$, which is the first axiom of (S_2)-system.

1 $p/CCqrCCpqCpr$, q/Nq $*C11$—70,

70 $CNqCCqrCCpqCpr$.

2 p/Nq, q/Cqr, $r/CCpqCpr$ $*C70$—$C5$ p/q, q/r—71,

71 $CNqCCpqCpr$.

23 p/Nq, q/Cpq, r/Cpr $*C71$—72,

72 $CCpqCNqCpr$.

Theses 56, 67, and 72 appear as three axioms of (S_2)-system.

Reference

[1] Y. Imai and K. Iséki: On axiom systems of propositional calculi, I. Proc. Japan Acad., **41**, 436-439 (1965).