112. *Boundary Convergence of Blaschke Products in the Unit-Circle. II*

By Chuji TANAKA
Mathematical Institute, Waseda University, Tokyo

(Comm. by Zyoiti SUETUNA, M.J.A., Sept. 13, 1965)

1. **Introduction.** We denote by $B(z)$ a Blaschke product in the unit circle:

$$B(z)=\prod_{n=1}^{+\infty} b(z, a_n)=\prod_{n=1}^{+\infty}\{1+c(z, a_n)\},$$

where $b(z, a)=\bar{a}/|a|\cdot(a-z)/(1-\bar{a}z)$, $0<|a_n|<1$, $S=\sum_{n=1}^{+\infty}(1-|a_n|)<+\infty$. For the sake of convenience, we make here a list of notations, which are used often in the sequel:

[1] $d(z, a)=(1-|a|^2)/(\bar{a}z-1)$.

[2] $c(z, a)=(1-|a|)/|a|+1/|a|\cdot d(z, a)$.

[3] $b(z, a)=1/|a|\cdot(1+d(z, a))$
$=1+c(z, a)$.

[4] $\theta_n=\arg b(1, a_n)$, $|\theta_n|\leqq\pi$.

[5] $r_n=|d(1, a_n)|=(1-|a_n|^2)/|1-a_n|$,
$R_n=(1-|a_n|)/|1-a_n|$.

[6] $\varphi_n=\arg d(1, a_n)=\arg b_n$, where $a_n=1+b_n$, $\pi/2<|\varphi_n|\leqq\pi$.

The object of this note is to establish some new theorems on boundary convergence of $B(z)$. Our main theorems read as follows:

Theorem 1.

(1.1) $\sum_{n=1}^{+\infty} R_n<+\infty$, *if and only if the following conditions hold simultaneously*:

$$(1.2)\qquad \begin{aligned}&(1)\quad \sum_{n=1}^{+\infty}|\theta_n|<+\infty,\\ &(2)\quad \lim_{n\to+\infty} R_n=0.\end{aligned}$$

Remark 1. (1) By the inequalities:

$$|c(1,a_n)|-(1-|a_n|)/|a_n|\leqq R_n\cdot(1+1/|a_n|)\leqq|c(1,a_n)|+(1-|a_n|)/|a_n|$$

$\sum_{n=1}^{+\infty}|c(1, a_n)|<+\infty$ *is equivalent to* $\sum_{n=1}^{+\infty} R_n<+\infty$ ([4] p. 67).

(2) In connection with Theorem 1, the following theorem due to O. Frostman ([2] p. 2) is very interesting; *The necessary and sufficient condition that $B(z)$ and all its partial products have the radial limit of modulus one at $z=1$ is that* $\sum_{n=1}^{+\infty} R_n<+\infty$.

Theorem 2 and 3 give the necessary and sufficient conditions for $\sum_{n=1}^{+\infty} c(1, a_n)$ to be convergent.

Theorem 2. *$\sum_{n=1}^{+\infty} c(1, a_n)$ is convergent, if and only if the following conditions hold simultaneously:*

$$(1.3)\qquad \begin{aligned} &(1)\quad \sum_{n=1}^{+\infty}(R_n)^2<+\infty \\ &(2)\quad \sum_{n=1}^{+\infty} R_n\delta_n \text{ is convergent, where} \end{aligned}$$

$$(1.4)\qquad \delta_n=\mathrm{sgn}\,[\sin(\varphi_n)].$$

Theorem 3. *$\sum_{n=1}^{+\infty} c(1, a_n)$ is convergent, if and only if the following conditions hold simultaneously:*

$$(1.5)\qquad \begin{aligned} &(1)\quad \sum_{n=1}^{+\infty}\theta_n^2<+\infty \\ &(2)\quad \sum_{n=1}^{+\infty} \theta_n \text{ is convergent.} \end{aligned}$$

Remark 2. If $\sum_{n=1}^{+\infty} c(1, a_n)$ is convergent, then $B(1)=\prod_{n=1}^{+\infty} b(1, a_n)$ is convergent, because $\prod_{n=1}^{+\infty} b(1, a_n)=\exp\left\{i\sum_{n=1}^{+\infty}\theta_n\right\}$ is convergent by (1.5) (2).

Theorem 4 is of Abelian type.

Theorem 4. *Suppose that $\sum_{n=1}^{+\infty} c(1, a_n)$ is convergent. Then $\lim_{r\to1} B(r)=B(1)$,*) if and only if*

$$(1.6)\qquad \lim_{r\to1-0}\sum_{n=1}^{+\infty} c(r, a_n)=\sum_{n=1}^{+\infty} c(1, a_n).$$

2. Theorem 1. We begin with the following inequalities:

$$(2.1)\qquad \begin{aligned} &(1)\quad |\theta_n|<\pi R_n \qquad \text{for } |\theta_n|\leqq\pi/2. \\ &(2)\quad |\theta_n|>\cos\delta\cdot R_n \qquad \text{for } \pi/2<|\varphi_n|\leqq\pi/2+\delta<\pi. \end{aligned}$$

Indeed, since $b(1, a_n)=1/|a_n|(1+d(1, a_n))$, we have easily

$$\theta_n=\arg\{1+d(1, a_n)\},\qquad |a_n|=|1+d(1, a_n)|,$$

so that, taking account of $d(1, a_n)=r_n\cdot\exp(i\varphi_n)$, by sine rule

$$(2.2)\qquad \sin\theta_n/r_n=\sin\varphi_n/|a_n|=\sin(\varphi_n-\theta_n)/1.$$

Since $2/\pi\cdot|\theta_n|\leqq|\sin\theta_n|$ for $|\theta_n|\leqq\pi/2$, by (2.2) $2|\theta_n|/\pi r_n\leqq1$, i.e.

$$(2.3)\qquad |\theta_n|<\pi R_n \qquad \text{for } |\theta_n|\leqq\pi/2.$$

By (2.2) $|\theta_n|/r\geqq\sin(\pi/2+\delta)/|a_n|=\cos\delta\cdot1/|a_n|$ for $\pi/2<|\varphi_n|\leqq\pi/2+\delta<\pi$, so that

$$(2.4)\qquad |\theta_n|>\cos\delta\cdot R_n \qquad \text{for } \pi/2<|\varphi_n|\leqq\pi/2+\delta<\pi.$$

By (2.3) and (2.4), (2.1) is completely established.

Suppose that (1.1) holds good. Then we have evidently $\lim_{n\to+\infty} R_n=0$, i.e. $\lim_{n\to+\infty} r_n=0$, so that by (2.2) $\lim_{n\to+\infty}\theta_n=0$. Hence there exists N such that $|\theta_n|\leqq\pi/2$ for $n\geqq N$. By (2.1)(1)

$$\sum_{n=N}^{+\infty}|\theta_n|<\pi\sum_{n=N}^{+\infty} R_n<+\infty.$$

*) By Remark 2, $B(1)$ exists.

Therefore (1.2) (1) and (2) follow from (1.1).

Next suppose that (1.2) (1) and (2) hold good. We divide the unit disk into the following three parts:

$$\Delta_1=(z;\ |z|<1,\ |z-1|\geqq\rho) \qquad (0<\rho<1)$$
$$\Delta_2=(z;\ |z|<1,\ |z-1|<\rho,\ |\arg(z-1)|\leqq\theta<\pi/2) \qquad (2\cos\theta=\rho),$$
$$\Delta_3=(z;\ |z|<1,\ |z-1|<\rho,\ |\arg(z-1)|>\theta).$$

Then, by (2.1) (2)

$$\sum_{a_n\in\Delta_3} R_n<\sec\delta\cdot\sum_{a_n\in\Delta_3}|\theta_n|<+\infty, \tag{2.5}$$

where $\delta=\pi/2-\theta$. We have evidently

$$\sum_{a_n\in\Delta_1} R_n\leqq 1/\rho\cdot\sum_{a_n\in\Delta_1}(1-|a_n|)<S/\rho<+\infty. \tag{2.6}$$

By the assumptions: $\lim_{n\to+\infty} R_n=0$, $z=1$ is not the limit point of $\{a_n\}$ contained in Δ_2. Hence possible limit points of $\{a_n\}$ contained in Δ_2 are the intersection points of $|z|=1$ and $|z-1|=\rho$ so that, denoting by ρ^* a sufficiently small positive constant, we have

$$\sum_{a_n\in\Delta_2} R_n\leqq 1/\rho^*\cdot\sum_{a_n\in\Delta_2}(1-|a_n|)<S/\rho^*<+\infty. \tag{2.7}$$

By (2.5), (2.6), and (2.7), $\sum_{n=1}^{+\infty} R_n<+\infty$. Thus (1.1) follows from (1.2) (1) and (2).

Since $b(r, a_n)$ maps the segment: $0\leqq r\leqq 1$ onto the circular arc of the circle passing through two points $|a_n|$ and $1/|a_n|$ and orthogonal to $|z|=1$, $|\arg(b(r, a_n))|$ is non-decreasing function of r for $0\leqq r\leqq 1$. Hence $\sum_{n=1}^{+\infty}|\theta_n|<+\infty$ is equivalent to $\lim_{r\to 1-0}\sum_{n=1}^{+\infty}|\arg b(r, a_n)|<+\infty$, so that, by Theorem 1 we get

Corollary 1. (C. Tanaka [4] pp. 68–69) $\sum_{n=1}^{+\infty} R_n<+\infty$, *if and only if the following two conditions hold simultaneously*:

$$(1)\quad \lim_{r\to 1-0}\sum_{n=1}^{+\infty}|\arg b(r, a_n)|<+\infty$$
$$(2)\quad \lim_{n\to+\infty} R_n=0.$$

As another application of Theorem 1, we obtain

Corollary 2. *If* $\sum_{n=1}^{+\infty} R_n=+\infty$, *then* $\lim_{r\to 1-0}\sum_{n=1}^{+\infty}|\arg b(r, a_n)|=+\infty$.

Proof. By Theorem 1, the following two alternatives are possible:

$$(2.8)\qquad \begin{aligned} &(1)\quad \varlimsup_{n\to+\infty} R_n>0 \\ &(2)\quad \lim_{n\to+\infty} R_n=0, \quad\text{and}\quad \sum_{n=1}^{+\infty}|\theta_n|=+\infty. \end{aligned}$$

In case (2.8) (1), by (2.2) $\varlimsup_{n\to+\infty}|\theta_n|>0$, so that $\sum_{n=1}^{+\infty}|\theta_n|=+\infty$. Thus, in any case we have $\sum_{n=1}^{+\infty}|\theta_n|=+\infty$. Since $|\arg b(r, a_n)|$ is non-decreasing function of r for $0\leqq r\leqq 1$, we have easily $\lim_{r\to 1-0}\sum_{n=1}^{+\infty}|\arg b(r, a_n)|=+\infty$, which proves Corollary 2.

If $\mathfrak{J}(a_n)\geqq 0$, then $\arg b(r, a_n)\geqq 0$ for $0\leqq r\leqq 1$. Hence, by Corollary 2 we get.

Corollary 3. (G. T. Cargo [1] p. 5) *If* $\sum_{n=1}^{+\infty} R_n=+\infty$ *and* $\mathfrak{J}(a_n)\geqq 0$ $(n=1,2,\cdots)$, *then* $\arg\{B(r)\}$ *tends monotonously to* $+\infty$ *as* $r\to 1-0$.

3. Theorem 2-4. Proof of Theorem 2. Since

$$c(1, a_n)=(1-|a_n|)/|a_n|+1/|a_n|\cdot r_n\cdot\exp(i\varphi_n),$$

the convergence of $\sum_{n=1}^{+\infty} c(1, a_n)$ is equivalent to the convergence of $\sum_{n=1}^{+\infty} r_n/|a_n|\cdot\cos\varphi_n+i\sum_{n=1}^{+\infty} r_n/|a_n|\cdot\sin\varphi_n$.

By (2.2), we have easily

$$-\cos\varphi_n=1/2\{|1-a_n|+r_n\} \tag{3.1}$$

so that

$$-\sum_{n=1}^{+\infty} r_n/|a_n|\cdot\cos\varphi_n=1/2\left\{\sum_{n=1}^{+\infty} r_n^2/|a_n|+\sum_{n=1}^{+\infty}(1-|a_n|)(1+1/|a_n|)\right\}.$$

Hence the convergence of $\sum_{n=1}^{+\infty} r_n/|a_n|\cdot\cos\varphi_n$ is equivalent to the convergence of $\sum_{n=1}^{+\infty} r_n^2$.

Defining δ_n as in (1.4), by (3.1) we get easily

$$\begin{aligned} r_n/|a_n|\cdot|\delta_n-\sin\varphi_n| &= r_n/|a_n|\cdot\cos^2\varphi_n/(1+|\sin\varphi_n|)\\ &=\{r_n^3+2r_n^2\cdot|1-a_n|+(1-|a_n|^2)\cdot|1-a_n|\}1/(4|a_n|\cdot(1+|\sin\varphi_n|))\end{aligned}$$

for $\pi/2<|\varphi_n|<\pi$. Since $1/2<1/(1+|\sin\varphi_n|)<1$ for $\pi/2<|\varphi_n|<\pi$, $\sum_{n=1}^{+\infty} r_n^2<+\infty$ means $\sum_{n=1}^{+\infty} r_n/|a_n|\cdot|\delta_n-\sin\varphi_n|+<\infty$, so that under the assumptions: $\sum_{n=1}^{+\infty} r_n^2<+\infty$, the convergence of $\sum_{n=1}^{+\infty} r_n/|a_n|\cdot\sin\varphi_n$ is equivalent to the convergence of $\sum_{n=1}^{+\infty}\delta_n r_n/|a_n|$.

Thus we have proved that the convergence of $\sum_{n=1}^{+\infty} c(1, a_n)$ is equivalent to the convergence of the following two series;

$$\text{(3.2)}\qquad \begin{aligned} &(1)\quad \sum_{n=1}^{+\infty} r_n^2,\\ &(2)\quad \sum_{n=1}^{+\infty}\delta_n r_n/|a_n|.\end{aligned}$$

Since $\sum_{n=1}^{+\infty} r_n^2=\sum_{n=1}^{+\infty} R_n^2\cdot(1+|a_n|)^2$

$$\sum_{n=1}^{+\infty}\delta_n r_n/|a_n|=2\sum_{n=1}^{+\infty}\delta_n R_n+\sum_{n=1}^{+\infty}(1-|a_n|)\cdot\delta_n R_n/|a_n|,$$

the convergence of (3.2) (1)–(2) is equivalent to (1.3) (1)–(2).

Proof of Theorem 3. By (2.2), we get easily

$$\cos\theta_n=(1+|a_n|^2-r_n^2)/2|a_n|,$$

so that

$$r_n^2/4|a_n|-\sin^2(\theta_n/2)=1/4|a_n|\cdot(1-|a_n|)^2. \tag{3.3}$$

By (3.3), $\sum_{n=1}^{+\infty} r_n^2$ is equivalent to $\sum_{n=1}^{+\infty}\theta_n^2<+\infty$. Since $\sin\theta_n-\theta_n=$

$\theta_n^3/3!+0(\theta_n^5)$, the convergence of $\sum_{n=1}^{+\infty}\theta_n$ is equivalent to the convergence of $\sum_{n=1}^{+\infty}\sin\theta_n$, provided that $\sum_{n=1}^{+\infty}\theta_n^2<+\infty$. By (2.2), $\sin\theta_n= r_n/|a_n|\cdot\sin\varphi_n$. Hence, under the assumptions: $\sum_{n=1}^{+\infty}\theta_n^2<+\infty$, $\sum_{n=1}^{+\infty}r_n/|a_n|\cdot\sin\varphi_n$ and $\sum_{n=1}^{+\infty}\theta_n$ are equiconvergent.

By what is proved in the proof of Theorem 2, the convergence of $\sum_{n=1}^{+\infty}c(1, a_n)$ is equivalent to the convergence of two series: $\sum_{n=1}^{+\infty}r_n^2$ and $\sum_{n=1}^{+\infty}r_n/|a_n|\cdot\sin\varphi_n$. Thus Theorem 3 is completely established.

Proof of Theorem 4. By the inequality:

$$|1-\bar{a}_n r|\geqq r\,|a_n-1| \qquad \text{for } 0<r\leqq 1,$$
$$|d(r, a_n)|=(1-|a_n|)/|\bar{a}_n r-1|\leqq r_n/r<2r_n \qquad \text{for } 1/2\leqq r\leqq 1.$$

Therefore,

$$(3.4)\qquad |c(r, a_n)|^2\leqq 1/|a_n|^2\{(1-|a_n|)+|d(r, a_n)|\}^2$$
$$<1/|a_n|^2\{(1-|a_n|)^2+4r_n(1-|a_n|)+4r_n^2\}$$

for $1/2\leqq r\leqq 1$.

Let us put

$$(3.5)\qquad \log b(r, a_n)-c(r, a_n)=\log\{1+c(r, a_n)\}-c(r, a_n)$$
$$=e(r, a_n)\cdot c^2(r, a_n).$$

If $\lim_{n\to+\infty} r_n=0$, then by (3.4) and (3.5)

$$(3.6)\qquad \lim_{n\to+\infty} e(r, a_n)=-1/2 \text{ uniformly for } 1/2\leqq r\leqq 1.$$

Suppose now that $\sum_{n=1}^{+\infty}c(1, a_n)$ is convergent. Then by Theorem 2, (3.4) and (3.6), $\sum_{n=1}^{+\infty}e(r, a_n)\cdot c^2(r, a_n)$ is absolutely and uniformly convergent for $1/2\leqq r\leqq 1$. Hence

$$\lim_{r\to1-0}\sum_{n=1}^{+\infty}e(r, a_n)\cdot c^2(r, a_n)=\sum_{n=1}^{+\infty}e(1, a_n)\cdot c^2(1, a_n),$$

so that, by (3.5)

$$(3.7)\qquad \lim_{r\to1-0}\left\{\log B(r)-\sum_{n=1}^{+\infty}c(r, a_n)\right\}$$
$$=\log B(1)-\sum_{n=1}^{+\infty}c(1, a_n).$$

By (3.7) $\lim_{r\to1-0}B(r)=B(1)$, if and only if $\lim_{r\to1-0}\sum_{n=1}^{+\infty}c(r, a_n)=\sum_{n=1}^{+\infty}c(1,a_n)$, which is to be proved.

As its application, we obtain

Corollary 4. (C. Tanaka [3] p. 410, [4] p. 70) *If $B(z)$ is absolutely convergent at $z=1$, then $\lim_{r\to1-0}B(r)=B(1)$.*

Proof. Since $\sum_{n=1}^{+\infty}|c(1, a_n)|<+\infty$, by Remark 1(1), $\sum_{n=1}^{+\infty}R_n<+\infty$, i.e. $\sum_{n=1}^{+\infty}r_n<+\infty$. Similarly as in (3.4), we have

$$|c(r, a_n)|<1/|a_n|\{(1-|a_n|)+2r_n\} \qquad \text{for } 1/2\leqq r\leqq 1,$$

so that $\sum_{n=1}^{+\infty} c(r, a_n)$ is absolutely and uniformly convergent for $1/2\leqq r\leqq 1$. Hence $\lim_{r\to 1-0} \sum_{n=1}^{+\infty} c(r, a_n)=\sum_{n=1}^{+\infty} c(1, a_n)$. Thus, by Theorem 4, Corollary 4 is proved.

Acknowledgement. The author wishes to express his hearty thanks to Prof. G. T. Cargo for his valuable criticism on [4] (MR 29 # 2399), which motivated this note.

References

[1] G. T. Cargo: The boundary behavior of Blaschke products. Jour. Math. Anal. App., **5**, 1–16 (1962).

[2] O. Frostman: Sur les produits de Blaschke. Kungl. Fysiogr. Sällsk. i Lund Förh., **12** (15), 169–182 (1942).

[3] C. Tanaka: Boundary convergence of Blaschke products in the unit-circle. Proc. Japan Acad., **39** (7), 410–412 (1963).

[4] ——: On Blaschke products in the unit-circle. Mem. School Sci. Engrg. Waseda Univ. Tokyo., **27**, 67–77 (1963).