140. *On Lacunary Fourier Series*

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Our first theorem is as follows:

Theorem 1. If the function f has the Fourier series

$$f(x)\sim\sum_{k=1}^{\infty}(a_{n_k}\cos n_kx+b_{n_k}\sin n_kx) \tag{1}$$

where

$$n_{k+1}-n_k>An_k^{\beta}\ (A \text{ constant and } 0<\beta\leqq 1) \tag{2}$$

and if f satisfies the α-Lipschitz condition ($\alpha>0$) at a point x_0, that is,

$$|f(x_0+t)-f(x_0)|\leqq A|t|^{\alpha} \quad \text{as} \quad t\to 0,$$

then we have

$$a_{n_k}=O(1/n_k^{\alpha\beta}),\quad b_{n_k}=O(1/n_k^{\alpha\beta}) \qquad (k=1,2,\cdots).$$

This is a generalization of theorems of Kennedy [1] and Tomić [2].

Proof. a) The case $1>\alpha>0$. We can suppose that $x_0=0$. Let c_{n_k} be the n_k-th complex Fourier coefficient of f, then

$$c_{n_k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-in_kx}dx.$$

We can suppose that[1)]

$$n_{k+1}-n_k\geqq An_k^{\beta} \quad \text{and} \quad n_k-n_{k-1}\geqq An_k^{\beta} \tag{2'}$$

and then we have

$$c_{n_k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)T_{M_k}(x)e^{-in_kx}dx$$

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1) If $\beta=1$, that is, $n_{k+1}/n_k\geqq\lambda>1$, then we can take $A=(\lambda-1)/\lambda$. In the case $0<\beta<1$, we can suppose that $n_{k+1}\geqq 2n_k$. For, if not, that is, if $n_{k+1}-n_k\geqq A'n_k^{\beta}$ for a constant A' and $n_{k+1}>2n_k$, then we insert the term $c_{n_{k'}}e^{in_{k'}x}$ with $n_{k'}=n_k+A'n_k^{\beta}$, then

$$n_{k'}-n_k=A'n_k^{\beta},\quad n_{k+1}-n_{k'}=(n_{k+1}-n_k)-A'n_k^{\beta}\geqq n_k-A'n_k^{\beta}\geqq A'n_k^{\beta}$$

for large k. If, further, $n_{k+1}>2n_{k'}$, then we insert also the term $c_{n_{k''}}e^{in_{k''}x}$ with $n_{k''}=n_{k'}+A'(n_{k'})^{\beta}$. Thus proceeding we get the sequence $(n_k^{(\nu)};\ \nu=1,2,\cdots,j)$ such that

$$n_k<n_{k'}<n_{k''}<\cdots<n_k^{(j)}<n_{k+1}$$

and

$$n_{k+1}\leqq 2n_k^{(j)},\quad n_k^{(\nu+1)}\leqq 2n_k^{(\nu)}(\nu=1,2,\cdots,j-1),\ n_{k'}\leqq 2n_k,$$
$$n_k^{(\nu+1)}-n_k^{(\nu)}\geqq A'(n_k^{(\nu)})^{\beta}\ (\nu=1,2,\cdots,j-1),\ n_{k+1}-n_k^{(j)}\geqq A'(n_k^{(j)})^{\beta},\ n_{k'}-n_k\geqq A'n_k^{\beta}.$$

This procedure is possible for all sufficiently large k. Now, instead of f, consider the function $g(x)=f(x)+h(x)$ where $h(x)\sim\sum_{\nu,k}c_k^{(\nu)}e^{in_k^{(\nu)}x}=\sum d_ke^{im_kx}$. We can take $(c_k^{(\nu)})$ such that h is sufficiently smooth. Then g satisfies the condition of f and the Fourier exponents (m_k) of g satisfy (2′) with $A=A'/2^{\beta}$.

for all trigonometrical polynomial $T_{M_k}(x)$ of degree $M_k \leqq An_k^\beta$ and with constant term 1. We take $T_{M_k}(x)$ as the twice of the Fejér kernel, that is,

$$T_{M_k}(x)=2K_{M_k}(x)=\frac{\sin^2 (M_k+1)x/2}{(M_k+1)\sin^2 x/2}.$$

Now

$$\begin{aligned}
c_{n_k}&=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)T_{M_k}(x)e^{-in_kx}dx=-\frac{1}{2\pi}\int_{-\pi}^{\pi} f\Big(x+\frac{\pi}{n_k}\Big)T_{M_k}\Big(x+\frac{\pi}{n_k}\Big)e^{-in_kx}dx\\
&=\frac{1}{4\pi}\int_{-\pi}^{\pi}\Big[f(x)T_{M_k}(x)-f\Big(x+\frac{\pi}{n_k}\Big)T_{M_k}\Big(x+\frac{\pi}{n_k}\Big)\Big]e^{-in_kx}dx\\
&=\frac{1}{4\pi}\int_{-\pi}^{\pi}\Big[f(x)-f\Big(x+\frac{\pi}{n_k}\Big)\Big]T_{M_k}(x)e^{-in_kx}dx\\
&\quad+\frac{1}{4\pi}\int_{-\pi}^{\pi} f\Big(x+\frac{\pi}{n_k}\Big)\Big[T_{M_k}(x)-T_{M_k}\Big(x+\frac{\pi}{n_k}\Big)\Big]e^{-in_kx}dx=I+J
\end{aligned}$$

where $J=0$, since the Fourier exponents of $f(x+\pi/n_k)$ with non-vanishing Fourier coefficients are the same as those of $f(x)$ and trigonometrical polynomial $T_{M_k}(x)-T_{M_k}(x+\pi/n_k)$ does not contain the constant term and is of order M_k. Therefore

$$\begin{aligned}
c_{n_k}&=I=\frac{1}{4\pi}\int_{-\pi}^{\pi}\Big[f(x)-f\Big(x+\frac{\pi}{n_k}\Big)\Big]T_{M_k}(x)e^{-in_kx}dx\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[f(x)-f\Big(x+\frac{\pi}{n_k}\Big)\Big]K_{M_k}(x)e^{-in_kx}dx\\
&=\frac{1}{2\pi}\Big(\int_{-\delta_k}^{\delta_k}+\int_{\delta_k}^{\pi}+\int_{-\pi}^{-\delta_k}\Big)\Big[f(x)-f\Big(x+\frac{\pi}{n_k}\Big)\Big]K_{M_k}(x)e^{-in_kx}dx=I_1+I_2+I_3
\end{aligned}$$

where $\delta_k=1/M_k$. We have

$$|I_1|\leqq\frac{M_k}{2\pi}\int_{-\delta_k}^{\delta_k}|f(x)-f(x+\pi/n_k)|\,dx=O(1/M_k^\alpha)=O(1/n_k^{\alpha\beta}),$$

$$|I_2|\leqq\frac{1}{2\pi M_k}\int_{\delta_k}^{\pi}|f(x)-f(x+\pi/n_k)|\frac{dx}{x^2}\leqq\frac{A}{M_k}\int_{\delta_k}^{\pi}\frac{dx}{x^{2-\alpha}}\leqq\frac{A}{n_k^{\alpha\beta}}$$

and I_3 may be estimated similarly as I_2. Thus the theorem is proved.

b) The case $\alpha\geqq 1$. In this case we use the polynomial

$$T_{M_k}(x)=(2K_{M_k/l}(x))^l\Big/\int_{-\pi}^{\pi}(2K_{M_k/l}(x))^l dx$$

instead of the Fejér kernel where l is a fixed integer depending on α, then we have

$$|T_{M_k}(x)|\leqq AM_k \quad\text{and}\quad |T_{M_k}(x)|\leqq A/M_k^{2l-1}t^{2l}.$$

Therefore, in the estimation of I_1 and I_2, α may be greater than or equal to 1. Thus the theorem holds for any $\alpha\geqq 1$.

Corollary 1. If f satisfies the α-Lipschitz condition at a point $(0<\alpha<1)$ and f has the Fourier series with the Hadamard gap, then f belongs to the Lip α class in the interval $(0, 2\pi)$.

Proof. By our theorem, the n_k-th Fourier coefficient of f is of order $O(1/n_k^\alpha)$ and then

$$|f(x+h)-f(x)| \leqq A \sum_{k=1}^{\infty} n_k^{-\alpha} \left| \sin \frac{1}{2} n_k h \right| \quad \text{for all } x \text{ and all } h.$$

If h is sufficiently small, then there is an m such that $1/n_{m+1} \leqq |h| \leqq 1/n_m$. We have

$$\sum_{k=1}^{m} n_k^{-\alpha} \left| \sin \frac{1}{2} n_k h \right| \leqq \frac{h}{2} \sum_{k=1}^{m} n_k^{1-\alpha} \leqq A |h| n_m^{1-\alpha} \leqq A |h|^\alpha$$

and

$$\sum_{k=m+1}^{\infty} n_k^{-\alpha} \left| \sin \frac{1}{2} n_k h \right| \leqq \sum_{k=m+1}^{\infty} n_k^{-\alpha} \leqq A n_{k+1}^{-\alpha} \leqq A |h|^\alpha.$$

Hence f belongs to the class Lip α in the whole interval.

Corollary 2. If f satisfies the 1-Lipschitz condition at a point and f has the Fourier series with the Hadamard gap, then $f \varepsilon \wedge$, that is,

$$f(x+h)-2f(x)+f(x-h)=O(|h|) \quad \text{for all } x.$$

Proof is similar as Corollary 1.

Theorem 2.[2] Let f satisfy the condition of Theorem 1 with $0<\beta<1$, then the Fourier series of f converges absolutely when $\alpha > \min(1/2\beta,\ 1/\beta-1)$.

Proof. a) Suppose that $\alpha > \beta^{-1}-1$. We shall prove that

(3) $n_j > Bj^\gamma$ for all sufficiently large j, a constant B and for any $\gamma < 1/(1-\beta)$. If we assume that $n_k > B\cdot k^\gamma$ for a k and for a B, $0<B<1$, then

$$n_{k+1} \geqq n_k + An_k^\beta \geqq Bk^\gamma + AB^\beta k^{\beta\gamma} \geqq B(k^\gamma + Ak^{\beta\gamma})$$
$$\geqq B(k^\gamma + \gamma k^{\gamma-1} + \cdots) = B(k+1)^\gamma \quad \text{for} \quad k \geqq k_0,$$

where k_0 is determined independently of B. We can take B, $0<B<1$ such that $n_{k_0} \geqq Bk_0^\gamma$. Thus we have $n_j > Bj^\gamma$ for all $j \geqq k_0$. We have now

$$\sum_{k=1}^{\infty} |c_{n_k}| \leqq A \sum_{k=1}^{\infty} \frac{1}{n_k^{\alpha\beta}} \leqq A \sum_{k=1}^{\infty} \frac{1}{k^{\alpha\beta\gamma}}$$

which is finite when $\alpha\beta\gamma > 1$. γ may be taken so near to $1/(1-\beta)$ such that $\alpha\beta\gamma > 1$ when $\alpha > \beta^{-1}-1$.

b) Suppose that $\alpha\beta > 1/2$. We suppose that $x_0=0$. Let us put

$$f_k(x) = f(x+\pi/4n_k) - f(x-\pi/4n_k)$$

then

$$f_k(x) \sim \sum_j c_{n_j} (e^{in_j(x+\pi/4n_k)} - e^{in_j(x-\pi/4n_k)}) = 2i \sum c_{n_j} \sin \frac{n_j \pi}{4n_k} e^{in_j x}.$$

If $T_{M_k}(x)$ is a trigonometrical polynomial of order An_k^β and with constant term 1, then the Fourier exponents with non-vanishing coefficients of $T_{M_k}(x) f_k(x)$ in the interval $(n_k, 2n_k)$ are the same as

2) This is the joint work of Mr. J.A. Chao and us.

those of $f_k(x)$ in the same interval. Thus we have

$$4\sum_{n_k\leqq n_j\leqq 2n_k}|c_{n_j}|^2\sin^2\frac{n_j\pi}{4n_k}\leqq\frac{1}{\pi}\int_{-\pi}^{\pi}f_k^2(x)T_{M_k}^2(x)dx$$

and hence

$$\sum_{n_k\leqq n_j\leqq 2n_k}|c_{n_j}|^2\leqq\frac{1}{4\pi}\int_{-\pi}^{\pi}f_k^2(x)T_{M_k}^2(x)dx.$$

We take $T_{M_k}(x)=2K_{M_k}(x)$, then the right side integral is, except for a factor 4,

$$\int_{-\pi}^{\pi}f_k^2(x)K_{M_k}^2(x)dx=\int_{-1/M_k}^{1/M_k}+\int_{1/M_k}^{\pi}+\int_{-\pi}^{-1/M_k}$$
$$\leqq 2\int_0^{1/n_k}\frac{M_k^2}{n_k^{2\alpha}}dx+2\int_{1/n_k}^{1/M_k}x^{2\alpha}M_k^2dx+\frac{2}{M_k^2}\int_{1/M_k}^{\pi}\frac{dx}{x^{2(2-\alpha)}}$$
$$\leqq\frac{A}{M_k^{2\alpha-1}}\leqq\frac{A}{n_k^{(2\alpha-1)\beta}}$$

and then

$$\sum_{n_k\leqq n_j\leqq 2n_k}|c_{n_j}|^2\leqq\frac{A}{n_k^{(2\alpha-1)\beta}},$$

$$\sum_{j=1}^{\infty}|c_j|\leqq\sum_{k=1}^{\infty}\sum_{2^k\leqq n_j\leqq 2^{k+1}}|c_{n_j}|\leqq A\sum_{k=1}^{\infty}\sqrt{2^{(1-\beta)k}\sum_{2^k\leqq n_j\leqq 2^{k+1}}|c_{n_j}|^2}$$
$$\leqq A\sum_{k=1}^{\infty}\frac{2^{(1-\beta)k/2}}{2^{(\alpha-1/2)\beta k}}=A\sum_{k=1}^{\infty}\frac{1}{2^{(\alpha\beta-1/2)k}}<\infty,$$

when $\alpha\beta>1/2$.

References

[1] P. B. Kennedy: Fourier series with gaps. Quart. J. Math. (Oxford), **7**, 224-230 (1956).

[2] M. Tomić: On the order of magnitude of Fourier coefficients with Hadamard's gaps. J. London Math. Soc., **37**, 117-120 (1962).