176. *Algebraic Formulations of Propositional Calculi*

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In this note, we shall concern with the Frege (F)-system and the Lukasiewicz (L_3)-system. As well known, the (L_3)-system:

1 *CpCqp*,

2 *CCpCqrCCpqCpr*,

3 *CCNpNqCqp*

characterizes two valued classical propositional calculus. In the (F)-system, the third axiom *CCNpNqCqp* are replaced into three axioms: *CCpqCNqNp*, *CNNpp*, and *CpNNp* and these five axioms give a complete axiom system for two valued propositional calculus.

If we take three axioms:

1 *CpCqp*,

2 *CCpCqrCCpqCpr*,

3′ *CCpNqCqNp*,

we can deduce *Cpp* and *CCpqCNqNp*. As already shown in [1] and [2], from axioms 1 and 2, we have

4 *Cpp*,

5 *CCpqCCqrCpr*,

and

6 *CCqrCCpqCpr*.

Then we have the following theses:

3′ *p*/*Nq* **C*4 *p*/*q*—7,

7 *CqNNq*.

6 *r*/*NNq* **C*7—8,

8 *CCpqCpNNq*.

5 *p*/*Cpq*, *q*/*CpNNq*, *r*/*CNqNp* **C*8—*C*3′ *q*/*Nq*—9,

9 *CCpqCNqNp*.

On the other hand, if we take

1 *CpCqp*,

2 *CCpCqrCCpqCpr*,

3″ *CCNpqCNqp*.

From the remark above, we have the theses 4, 5, and 6 by the axioms 1 and 2. Further we have the following theses by the same techniques above:

3″ *q*/*Np* **C*4—7,

7 *CNNpp*.

5 *p*/*NNp*, *q*/*p*, *r*/*q* **C*7—8,

8 *CCpqCNNpq*.

5 p/Cpq, $q/CNNpq$, $r/CNqNp$ $*C8—C3''$ $p/Np—9$,
9 $CCpqCNqNp$.

Therefore, under the axioms 1 and 2, we have

$$\begin{tikzcd} & (3') \arrow[dr, Rightarrow] & \\ (3) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (9) \\ & (3'') \arrow[ur, Rightarrow] & \end{tikzcd}$$

In these systems, from the table

	0	1	N
0	0	1	0
1	0	0	1

we know that the first two axioms are independent to each axiom containing the symbol N.

Consider the following table:

	0	1	N
0	0	1	0
1	0	0	0

Then $CCpqCNqNp$ and $CCpNqCqNp$ always have the designated value 0. On the other hand, if we substitute in the thesis $CCNpqNqp$ for $p/1$ and $q/0$, we have $CCN10CN01=CC00C01=C01=1$.

Next we shall consider the following table:

	0	1	N
0	0	1	1
1	0	0	1

Then $CCpqCNqNp$ and $CCNpqCNqp$ have the designated value 0. In the thesis $CCpNqCqNp$, we substitute for $p/1$ and $q/0$, then $CCN1N0C01=CC111=C01=1$. Therefore from the results above, we have the following

Theorem 1. *The axioms* 1, 2 *and* $CCpqNqNp$ *do not imply* $CCNpqCNqp$, $CCpNqCqNp$, *and* $CCNpNqCqp$. *The axioms* 1, 2 *and* $CCNpqCNqp$ (*or* $CCpNqCqNp$) *do not imply* $CCpNqCqNp$ (*or* $CCNpqCNqp$) *and* $CCNpNqCqp$.

By this fundamental theorem, we have three algebraic systems as well as the usual Boolean algebra. From these considerations, we have algebraic systems by the following technique (see an algebraic formulation of positive logic, see [3]).

Let $\langle X, 0, *, \sim\rangle$ be an algebra consisting of a set X containing an element 0 and a binary operation $*$ and an unary operation $\sim$ defined on X such that the axioms given below hold. For convenient, we introduce an order relation. We write $x\leqslant y$ for $x*y=0$. Then the axioms are written as follows. For $x, y, z \in X$,

1　$x*y\leqslant x$.
2　$(x*z)*(y*z)\leqslant(x*y)*z$.
3　$x*y\leqslant(\sim y)*(\sim x)$.
4　$0\leqslant x$.
5　$x\leqslant y$ and $y\leqslant x$ imply $x=y$.

Here we can replace the axiom 3 into other axioms:

3′　$(\sim x)*y\leqslant(\sim y)*x$,
3″　$x*(\sim y)\leqslant y*(\sim x)$
or
3‴　$(\sim x)*(\sim y)\leqslant y*x$.

Therefore we have four algebraic systems.[1] From axiom 3′, we have

$$(\sim y)*x\leqslant(\sim x)*y\ ,$$

which shows that $(\sim x)*y=(\sim y)*x$ by axiom 5. Similarly from the axioms 3″ and 5, we have $x*(\sim y)=y*(\sim x)$.

Theorem 2. *Under axioms* 3′ *and* 5, *we have*

$$(\sim x)*y=(\sim y)*x$$

and under axioms 3″, 5, *we have*

$$x*(\sim y)=y*(\sim x).$$

If we add an axiom $x=\sim(\sim x)$, then from axiom 3, we have

$$(\sim x)*(\sim y)\leqslant(\sim(\sim y))*(\sim(\sim x))=y*x.$$

Hence $(\sim y)*(\sim x)\leqslant x*y$. Axioms 3, 5 and the relation obtained imply $x*y=(\sim y)*(\sim x)$. Therefore we have the following

Theorem 3. *If, in each one of algebraic systems above, we add* an axiom $x=\sim(\sim x)$, *we have* $x*y=(\sim y)*(\sim x)$.

On the other hand, from axioms 1, 2, 4, and 5, we have the relations between $*$ and $\leqslant$, as already shown by L. Henkin [3]. These relations are fundamental for developing our theory. We shall prove some of them.

1) $0*x=0$.

In axiom 1, we substitute $x/0$ and y/x, then $0*x\leqslant 0$. Hence by axiom 5, we have $0*x=0$.

2) $x*x=0$, i.e. $x\leqslant x$.

By axiom 1, we have $(x*x)*x=0$ and $(x*(x*x))*x=0$. From axiom 2, $(x*x)*((x*x)*x)\leqslant(x*(x*x))*x$. This shows $(x*x)*0\leqslant 0$ by axioms 4 and 5.

3) *If* $x*y=0$, $y*z=0$, *then* $x*z=0$, i.e. $x\leqslant y$, $y\leqslant z$ *imply* $x\leqslant z$.

By axiom 2 and proposition 1), we have $(x*z)*0\leqslant 0*z=0$. Hence $x*z=0$, i.e. $x\leqslant z$.

As a result corresponding the *commutative transportation law* in propositional calculi, we have

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1) The fundamental properties of equality are obtained by propositions 2 and 3 below.

4) *If* $x*y\leqslant z$, *then* $x*z\leqslant y$, i.e. $(x*y)*z=0$ *implies* $(x*z)*y=0$.

Axiom 2 and $(x*y)*z=0$ imply $(x*z)*(y*z)=0$. By $y*z\leqslant y$, we have $x*z\leqslant y$, i.e. $(x*z)*y=0$.

On the logical *syllogistic law*, we have the following

5) *If* $x\leqslant y$, *then* $z*y\leqslant z*x$, i.e. $x*y=0$ *implies* $(z*y)*(z*x)=0$.

Axioms 1 and 2 mean $(z*x)*y\leqslant z*x$ and $(z*y)*(x*y)\leqslant(z*x)*y$ respectively. By $x*y=0$ and the inequalities above, we have $(z*y)*0\leqslant(z*x)*y\leqslant z*x$. Hence by proposition 4, $(z*y)*(z*x)=0$, i.e. $z*y\leqslant z*x$.

6) *If* $x\leqslant y$, *then* $x*z\leqslant y*z$, i.e. $x*y=0$ *implies* $(x*z)*(y*z)=0$.

From axiom 2 and proposition 1, we have $(x*z)*(y*z)\leqslant(x*y)*z=0*z=0$. Hence by axioms 4 and 5, we have $(x*z)*(y*z)=0$, which proves proposition 6.

7) $y*x=(y*x)*x$.

From axiom 2, we have $(y*x)*(x*x)\leqslant(y*x)*x$. Further, by proposition 2, $(y*x)*0\leqslant(y*x)*x$, and $y*x\leqslant(y*x)*x\leqslant y*x$ by the commutative law and axiom 1.

Now consider an abstract algebra $\langle X, 0, *, \sim\rangle$.

Definition. *If* $\langle X, 0, *, \sim\rangle$ *satisfies axioms* 1, 2, 3 (3′, 3″, 3‴), 4, *and* 5, *it is called a B* (*NB*, *BN*, *NBN*)-*algebra respectively.*

Consider an *NB*-algebra $\langle X, 0, *, \sim\rangle$, then $(\sim x)*(\sim x)=(\sim(\sim x))*x$ by $(\sim x)*y=(\sim y)*x$. From proposition 2, $(\sim x)*(\sim x)=0$, therefore we have $\sim(\sim x)\leqslant x$. By theorem 2, we have $(\sim x)*y=(\sim y)*x$. Hence we have $(\sim x)*(\sim y)=\sim(\sim y)*x\leqslant y*x$ by proposition 6. This shows that any *NB*-algebra is an *NBN*-algebra. Similarly, for a *BN*-algebra, we have $x\leqslant\sim(\sim x)$, and from theorem 2, $x*(\sim y)=y*(\sim x)$, hence we have $(\sim x)*(\sim y)=y*(\sim(\sim x))\leqslant y*x$ by proposition 5. Therefore, we have the following

Theorem 4. *Any BN-algebra* (*NB-algebra*) *is an NBN-algebra.*

This corresponds to theorem 1. Next we shall prove that any *B*-algebra is an *NB*-algebra and a *BN*-algebra. To prove it we need some propositions.

8) $x*y\leqslant\sim y$, $x*(\sim y)\leqslant y$.

By axioms 1 and 3, we have

$$x*y\leqslant(\sim y)*(\sim x)\leqslant\sim y$$

and moreover, by the commutative law 4, we have $x*(\sim y)\leqslant y$.

9) $x\leqslant\sim(\sim x)$.

By axiom 3, we have

$$x*(\sim(\sim x))\leqslant\sim(\sim(\sim x))*(\sim x)\,.$$

On the other hand, in the first inequality of proposition 8, we substitute $\sim(\sim(\sim x))$ for x and $\sim x$ for y, then we have

$$\sim(\sim(\sim x))*(\sim x)\leqslant\sim(\sim x).$$

Hence by proposition 3, $x*(\sim(\sim x))\leqslant\sim(\sim x)$, i.e. $(x*(\sim(\sim x)))*(\sim(\sim x))=0$. Applying proposition 7, we have $x*(\sim(\sim x))=(x*(\sim(\sim x)))*(\sim(\sim x))=0$, which shows $x*(\sim(\sim x))=0$. This means $x\leqslant\sim(\sim x)$.

10) $\sim(\sim x)\leqslant x$.

By axiom 3 and $x*(\sim(\sim x))=0$, we have

$$(\sim(\sim x))*x\leqslant(\sim x)*(\sim(\sim(\sim x)))=0.$$

Hence $\sim(\sim x)\leqslant x$.

From 9 and 10, we have

11) $\sim(\sim x)=x$.

By axiom 3 and proposition 3, we have

$$x*(\sim y)\leqslant(\sim(\sim y))*(\sim x)=y*(\sim x).$$

This means that any B-algebra is a BN-algebra. Similarly, we have the following

Theorem 5. *Any B-algebra is a BN-algebra and an NB-algebra.* $\sim(\sim x)=x$ *holds in the B-algebra.*

References

1] Y. Arai: On axiom systems of propositional calculi. III. Proc. Japan Acad., **41**, 570-574 (1965).

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3] L. Henkin: An algebraic characterization of quantifiers. Fund. Math., **37**, 63-74 (1950).