168. The Relation between (N, p_n) and $(\bar{N}, p_n)$ Summability. II

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§ 1. The present note is a continuation of the previous paper by the author [2]. We suppose, throughout this note,[1] that

$$p_n>0, \qquad \sum_{n=0}^{\infty} p_n=\infty,$$
$$P_n=p_0+p_1+\cdots+p_n, \; n=0, 1, \cdots.$$

The Nörlund transformation (N, p_n) is defined as transforming the sequence $\{s_n\}$ into the sequence $\{t_n\}$ by means of the equation

$$(1) \qquad t_n=\frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}s_\nu.$$

As is well known, this transformation is regular if

$$(2) \qquad \lim_{n\to\infty}\frac{p_n}{P_n}=0.$$

See Hardy [1], p. 64.

The discontinuous Riesz transformation $(\bar{N}, p_n)$ is defined as transforming the sequence $\{s_n\}$ into the sequence $\{u_n\}$ by means of the equation

$$(3) \qquad u_n=\frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu.$$

This transformation is regular (see Hardy [1], p. 57).

From (1) we see easily

$$\sum_{\nu=0}^{n} P_{n-\nu}s_\nu=\sum_{\nu=0}^{n} P_\nu t_\nu.$$

Thus we obtain the following

Theorem 1. *(N, P_n) is equivalent*[2] *to the iteration product $(\bar{N}, P_n)\cdot(N, p_n)$.*

§ 2. We shall prove here the following

Theorem 2. *If*

$$(4) \qquad \{p_n\} \text{ is non-increasing,}$$

and if

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1) In Lemma, we need not assume $\sum_{n=0}^{\infty} p_n=\infty$ generally.

2) Given two summability methods A, B, we say that A *implies* B if any series or sequence summable A is summable B to the same sum. We say that A and B are *equivalent* if A implies B and B implies A.

$$(5)\qquad \frac{p_{n+1}}{p_n} \geq \frac{p_n}{p_{n-1}}, \qquad n=1, 2, \cdots,$$

then (N, p_n) *implies* $(\bar{N}, p_n)$.

In order to prove the theorem, we require the following

Lemma. *If* $p(x)=\sum_{n=0}^{\infty} p_n x^n$ *is convergent for* $|x|<1$, *and if*

$$p_n>0, \qquad n=0, 1, \cdots,$$

$$\frac{p_{n+1}}{p_n} \geq \frac{p_n}{p_{n-1}}, \qquad n=1, 2, \cdots,$$

then

$$\{p(x)\}^{-1}=\frac{1}{p_0}+q_1x+q_2x^2+\cdots,$$

where

$$q_n \leq 0, \qquad n=1, 2, \cdots,$$

$$\sum_{n=1}^{\infty} |q_n| \leq \frac{1}{p_0}.$$

If $\sum_{n=0}^{\infty} p_n=\infty$, *then* $\sum_{n=1}^{\infty} |q_n|=\frac{1}{p_0}$.

For the proof of this lemma, see, e.g., Hardy [1], Theorem 22.

We now give the proof of our theorem. From (4) we see easily that

$$\lim_{n\to\infty} \frac{P_{n-1}}{P_n}=1,$$

that $\sum_{n=0}^{\infty} P_n x^n$ is convergent for $|x|<1$, and that $p(x)=\sum_{n=0}^{\infty} p_n x^n$ also converges for $|x|<1$. Since $p_0 \neq 0$, $q(x)=\{p(x)\}^{-1}=\sum_{n=0}^{\infty} q_n x^n$ has a non-zero radius of convergence. Now the transformation inverse to (1) is

$$(6)\qquad s_n=\sum_{k=0}^{n} q_{n-k} P_k t_k.$$

See Kuttner [3]. From (3) and (6) we obtain

$$(7)\qquad \begin{aligned} u_n &= \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \sum_{k=0}^{\nu} q_{\nu-k} P_k t_k \\ &= \frac{1}{P_n} \sum_{k=0}^{n} P_k t_k \sum_{\nu=0}^{n-k} p_{k+\nu} q_\nu \\ &= \sum_{k=0}^{n} b_{nk} t_k, \end{aligned}$$

where

$$b_{nk}=\frac{P_k}{P_n} \sum_{\nu=0}^{n-k} p_{k+\nu} q_\nu.$$

Now if $s_\nu=1$ for all ν, then $t_n=1$, $u_n=1$ for all n. Hence $\sum_{k=0}^{n} b_{nk}=1$ for all n. Also, since $P_n \to \infty$ and $q_n \to 0$, we see easily that $b_{nk} \to 0$ as $n \to \infty$ for any fixed k. Hence a necessary and sufficient condition for the transformation (7) to be regular is that

$$\sum_{k=0}^{n} |b_{nk}| = O(1). \tag{8}$$

Since

$$\begin{aligned} b_{nk} &= \frac{P_k}{P_n}(p_k q_0 + p_{k+1} q_1 + \cdots + p_n q_{n-k}) \\ &\geq \frac{P_k}{P_n}\left\{\frac{p_k}{p_0} - p_k(|q_1| + |q_2| + \cdots + |q_{n-k}|)\right\} \\ &\geq 0 \end{aligned}$$

from (4) and the lemma, we get (8).

This proves our assertion.

Combining the last theorem and Theorem 1 of the previous paper [2], we obtain the following

Theorem 3. *If* $\{p_n\}$ *is non-increasing, and if*

$$p_n \geq \sigma > 0, \qquad n = 0, 1, \cdots,$$

$$\frac{p_{n+1}}{p_n} \geq \frac{p_n}{p_{n-1}}, \quad n = 1, 2, \cdots,$$

then (N, p_n) *and* $(\bar{N}, p_n)$ *are equivalent.*

References

[1] G. H. Hardy: Divergent Series. Oxford (1949).

[2] K. Ishiguro: The relation between (N, p_n) and $(\overline{N}, p_n)$ summability. Proc. Japan Acad., **41**, 120-122 (1965).

[3] B. Kuttner: The high indices theorem for discontinuous Riesz means. Jour. London Math. Soc., **39**, 635-642 (1964).