204. Decompositions of Generalized Algebras. II

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Theorem 3. *Every genalgebra* $\mathfrak{S}=\langle G, o_1, \cdots, o_n, A\rangle$ *with finitary operations is isomorphic with a subdirect product of subdirectly irreducible genalgebras.*

Proof. Consider arbitrary elements $x, y \in G$, $a, b \in A$ such that $x \neq y$ and $a \neq b$. Let $\mathcal{L}(x, y; a, b)$ be the family of all reduced congruences (θ, φ) of $\mathcal{L}(x, y; a, b)$ such that

$$(x, y) \notin \theta \quad \text{and} \quad (a, b) \notin \varphi.$$

Since $(\Delta_G, \Delta_A) \in \mathcal{L}(x, y; a, b)$, then $\mathcal{L}(x, y; a, b) \neq \varnothing$. It is partially ordered and every linearly ordered subset of it possesses an upper bound given by its join. Hence, by Zorn's lemma, $\mathcal{L}(x, y; a, b)$ has a maximal element $(\theta_{xy}, \varphi_{ab})$. To show that the quotient genalgebra

$$\mathfrak{S}/(\theta_{xy}, \varphi_{ab}) = \langle G/\theta_{xy}, o_1, \cdots, o_n, A/\varphi_{ab}\rangle$$

is subdirectly irreducible, it suffices to show that it has no proper reduced congruences and hence no proper congruences. If it does possess proper reduced congruences, let $(\tilde{\theta}_\lambda, \tilde{\varphi}_\lambda)$ $(\lambda \in \Lambda)$ be the family of all reduced congruences in $\mathfrak{S}/(\theta_{xy}, \varphi_{ab})$. By Theorem C each such congruence $(\tilde{\theta}_\lambda, \tilde{\varphi}_\lambda)$ corresponds to a reduced congruence $(\theta_\lambda, \varphi_\lambda)$ in $\mathfrak{S}$ such that

$$(\theta_\lambda, \varphi_\lambda) \gneqq (\theta_{xy}, \varphi_{ab}).$$

Clearly, $\theta_\lambda \supsetneqq \theta_{xy}$ for all $\lambda \in \Lambda$; for, if $\theta_\lambda = \theta_{xy}$, then $\varphi_\lambda = \varphi_{ab}$, since both congruences are reduced. Thus we have $\bigcap_{\lambda \in \Lambda} \theta_\lambda \supsetneqq \theta_{xy}$ and in any case

$$\bigcap_{\lambda \in \Lambda} (\theta_\lambda, \varphi_\lambda) \gneqq (\theta_{xy}, \varphi_{ab}).$$

The reduction

$$\bigcap_{\lambda \in \Lambda} (\theta_\lambda, \varphi)$$

of the congruence on the left side must properly contain the congruence on the right side; for, if $\varphi \subsetneqq \varphi_{ab}$, then

$$(\bigcap_{\lambda \in \Lambda} \theta_\lambda, \varphi) \cap (\theta_{xy}, \varphi_{ab}) = (\bigcap_{\lambda \in \Lambda} \theta_\lambda \cap \theta_{xy}, \varphi \cap \varphi_{xy}) = (\theta_{xy}, \varphi)$$

contrary to the fact that $(\theta_{xy}, \varphi_{ab})$ is reduced. Whence the genalgebra $\mathfrak{S}/(\theta_{xy}, \varphi_{ab})$ is subdirectly irreducible. Obviously,

$$\bigcap_{x \neq y} \bigcap_{a \neq b} (\theta_{xy}, \varphi_{ab}) = (\bigcap_{x \neq y} \theta_{xy}, \bigcap_{a \neq b} \varphi_{ab}) = (\Delta_G, \Delta_A)$$

and therefore the final conclusion follows.

Theorem 4. *The necessary and sufficient conditions for a genalgebra* $\mathfrak{S}=\langle G, o_1, \cdots, o_n, A\rangle$ *to be isomorphic to a direct product of genalgebras* $\mathfrak{S}_\lambda = \langle G_\lambda, o_1^\lambda, \cdots, o_n^\lambda, A_\lambda\rangle (\lambda \in \Lambda)$ *are that* (1) *there exists* *for each* $\lambda \in \Lambda$ *a homomorphism* (h_λ, k_λ) *of* $\mathfrak{S}$ *onto* $\mathfrak{S}_\lambda$ *whose kernels* $(\theta_\lambda, \varphi_\lambda) = (h_\lambda h_\lambda^{-1}, k_\lambda k_\lambda^{-1})$ *satisfy the condition*

$$\bigcap_{\lambda \in \Lambda} (\theta_\lambda, \varphi_\lambda) = (\Delta_G, \Delta_A); \qquad \textit{and}$$

(2) *for each pair of subsets* $\{x_\lambda \mid \lambda \in \Lambda\} \subseteq G$ *and* $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq A$, *there exist elements* $x \in G$ *and* $a \in A$ *such that*

$$(x, x_\lambda) \in \theta_\lambda \qquad \textit{and} \qquad (a, a_\lambda) \in \varphi_\lambda$$

for all $\lambda \in \Lambda$.

Proof. If $\mathfrak{S}$ is isomorphic onto $\prod_{\lambda \in \Lambda} \mathfrak{S}_\lambda$ under (f, g), then clearly condition (1) holds. To prove condition (2), consider any $\{x_\lambda \mid \lambda \in \Lambda\} \subseteq G$. Let $\chi \in \prod_{\lambda \in \Lambda} \mathfrak{S}_\lambda$ such that $\chi(\lambda) = f(x_\lambda)(\lambda)$ for $\lambda \in \Lambda$. Let $x \in G$ such that $f(x) = \chi$. Then $f(x)(\lambda) = \chi(\lambda) = f(x_\lambda)(\lambda)$ and hence $h_\lambda(x) = p_\lambda f(x) = p_\lambda f(x_\lambda) = h_\lambda(x_\lambda)$ for $\lambda \in \Lambda$. Thus, $(x, x_\lambda) \in \theta_\lambda$ for all $\lambda \in \Lambda$. In an analogous manner, there exists an $a \in A$ such that $(a, a_\lambda) \in \varphi_\lambda$ for all $\lambda \in \Lambda$.

Conversely, suppose conditions (1) and (2) hold. By Theorem 1, $\mathfrak{S}$ is isomorphic to a subgenalgebra of the direct product $\prod_{\lambda \in \Lambda} \mathfrak{S}/(\theta_\lambda, \varphi_\lambda)$ under (f, g) such that

$$f(x)(\lambda) = x/\theta_\lambda \qquad \text{and} \qquad g(a)(\lambda) = a/\varphi_\lambda$$

where $\theta_\lambda = h_\lambda h_\lambda^{-1}$ and $\varphi_\lambda = k_\lambda k_\lambda$. Thus, it suffices to show that both f and g are onto. Let $\chi \in \prod_{\lambda \in \Lambda} \mathfrak{S}/(\theta_\lambda, \varphi_\lambda)$ such that

$$\chi(\lambda) = x_\lambda/\theta_\lambda \qquad \text{for} \quad \lambda \in \Lambda.$$

Corresponding to $\{x_\lambda \mid \lambda \in \Lambda\} \subseteq G$, by (2), there exists an element $x \in G$ such that $(x, x_\lambda) \in \theta_\lambda$, in other words, $x/\theta_\lambda = x_\lambda/\theta_\lambda$ for $\lambda \in \Lambda$. Whence $f(x) = \chi$. Similarly for g.

Corollary 5. *There exists a one-to-one correspondence between the direct product representations of a genalgebra* $\mathfrak{S}$ *and the collection of all sets of congruences* $\{(\theta_\lambda, \varphi_\lambda) \mid \lambda \in \Lambda\}$ *of* $\mathfrak{S}$ *satisfying the conditions*

(1) $\bigcap_{\lambda \in \Lambda} (\theta_\lambda, \varphi_\lambda) = (\Delta_G, \Delta_A)$;

(2) *for* $\{x_\lambda \mid \lambda \in \Lambda\} \subseteq G$ *and* $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq A$, *there are elements* $x \in G$ *and* $a \in A$ *such that*

$$(x, x_\lambda) \in \theta_\lambda \qquad \textit{and} \qquad (a, a_\lambda) \in \varphi_\lambda$$

for $\lambda \in \Lambda$.

Theorem 6. *A necessary and sufficient condition for a genalgebra* $\mathfrak{S}$ *to be directly reducible is that there exist two congruences* $(\theta_1, \varphi_1) \neq (\Delta_G, \Delta_A)$ *and* $(\theta_2, \varphi_2) \neq (\Delta_G, \Delta_A)$ *such that*

(1) $\theta_1\theta_2 = \theta_2\theta_1$ *and* $\varphi_1\varphi_2 = \varphi_2\varphi_1$;

(2) $(\theta_1, \varphi_1) \vee (\theta_2, \varphi_2) = (G \times G, A \times A)$;

(3) $(\theta_1, \varphi_1) \cap (\theta_2, \varphi_2) = (\Delta_G, \Delta_A)$.

Under these conditions

$$\mathfrak{S} \cong \mathfrak{S}/(\theta_1, \varphi_1) \times \mathfrak{S}/(\theta_2, \varphi_2).$$

Proof. First suppose $\mathfrak{S}$ is isomorphic to $\mathfrak{S}_1\times\mathfrak{S}_2$ under (f, g) such that $f(x)=(x_1, x_2)$ and $g(a)=(a_1, a_2)$, where the homomorphisms (h_λ, k_λ) defined by $h_\lambda(x)=x_\lambda$ and $k_\lambda(a)=a_\lambda$ are non-isomorphisms. This means $(\theta_\lambda, \varphi_\lambda)=(h_\lambda h_\lambda^{-1}, k_\lambda k_\lambda^{-1})\neq(\Delta_G, \Delta_A)$. Let $x, y\in G$ such that $f(x)=(x_1, x_2)$ and $f(y)=(y_1, y_2)$. Then

$$(f^{-1}(x_1, x_2), f^{-1}(x_1, y_2))\in\theta_1 \quad \text{and} \quad (f^{-1}(x_1, y_2), f^{-1}(y_1, y_2))\in\theta_2.$$

Hence $(x, y)=(f^{-1}(x_1, x_2), f^{-1}(y_1, y_2))\in\theta_1\theta_2$. Thus, $\theta_1\theta_2=G\times G$ and similarly $\theta_2\theta_1=G\times G$. Whence $\theta_1\theta_2=\theta_2\theta_1$ The same conclusion may be derived for φ_1 and φ_2. Therefore,

$$(\theta_1, \varphi_1)\vee(\theta_2, \varphi_2)=(\theta_1\vee\theta_2, \varphi_1\vee\varphi_2)=(\theta_1\theta_2, \varphi_1\varphi_2)=(G\times G, A\times A).$$

If $(x, y)\in\theta_1\cap\theta_2$, so that $(x, y)\in\theta_1$ and $(x, y)\in\theta_2$, then $x_1=y_1$ and $x_2=y_2$. Since f is one-to-one, then $x=y$ or $(x, y)\in\Delta_G$. Hence $\theta_1\cap\theta_2=\Delta_G$ and in an analogous manner, $\varphi_1\cap\varphi_2=\Delta_A$. The conclusion follows.

Conversely, suppose (θ_i, φ_i) $(i=1, 2)$ are non-trivial congruences satisfying (1) and (2). Consider the product genalgebra $\mathfrak{S}/(\theta_1, \varphi_1)\times\mathfrak{S}/(\theta_2, \varphi_2)=\langle G/\theta_1\times G/\theta_2, O_1, \cdots, O_n, A/\varphi_1\times A/\varphi_2\rangle$ and define $f\colon G\Rightarrow G/\theta_1\times G/\theta_2$ and $g\colon A\Rightarrow A/\varphi_1\times A/\varphi_2$ such that $f(x)=(x/\theta_1, x/\theta_2)$ and $g(a)=(a/\varphi_1, a/\varphi_2)$. The pair (f, g) is clearly a homomorphism, for, if $i=1, \cdots, n$ and $x_1, \cdots, x_{m_i}\in G$, then

$$\begin{aligned} g(o_i(x_1, \cdots, x_{m_i}))&=(o_i(x_1, \cdots, x_{m_i})/\varphi_1, o_i(x_1, \cdots, x_{m_i})/\varphi_2)\\ &=(O_i^1(x_1/\theta_1, \cdots, x_{m_i}/\theta_1), O_i^2(x_1/\theta_2, \cdots, x_{m_i}/\theta_2))\\ &=O_i((x_1/\theta_1, x_1/\theta_2), \cdots, (x_{m_i}/\theta_1, x_{m_i}/\theta_2))=O_i(f(x_1), \cdots, f(x_{m_i})). \end{aligned}$$

If $(x/\theta_1, x/\theta_2)=(y/\theta_1, y/\theta_2)$, so that $(x, y)\in\theta_1$ and $(x, y)\in\theta_2$, then $(x, y)\in\theta_1\cap\theta_2=\Delta_G$. Whence $x=y$. Finally, if $(x_1/\theta_1, x_2/\theta_2)\in G/\theta_1\times G/\theta_2$, then, inasmuch as $G\times G=\theta_1\vee\theta_2=\theta_1\theta_2$, there exists an $x\in G$ such that

$$(x, x_1)\in\theta_1 \quad \text{and} \quad (x, x_2)\in\theta_2.$$

Thus, $(x_1/\theta_1, x_2/\theta_2)=(x/\theta_1, x/\theta_2)=f(x)$. Therefore, f (and similarly g) is onto. The proof is now complete.

For convenience, let us call a family of congruences in a genalgebra *permutable* if and only if for each pair (θ_1, φ_1) and (θ_2, φ_2) of the family we have $\theta_1\theta_2=\theta_2\theta_1$ and $\varphi_1\varphi_2=\varphi_2\varphi_1$.

Theorem 7. *Let $\mathfrak{S}$ be a genalgebra with permutable congruences. Then $\mathfrak{S}$ is isomorphic with a direct product of the genalgebras $\mathfrak{S}_j=\langle G_j, o_1^j, \cdots, o_n^j, A_j\rangle$ $(j=1, \cdots, m)$ if and only if for each $j=1, \cdots, m$, there exists a homomorphism (h_λ, k_λ) of $\mathfrak{S}$ onto $\mathfrak{S}_j$ whose kernels $(\theta_j, \varphi_j)=(h_jh_j^{-1}, k_jk_j^{-1})$ satisfy the conditions*

(1) $\bigcap_{j=1}^{m}(\theta_j, \varphi_j)=(\Delta_G, \Delta_A)$;

(2) $\bigcap_{j=1}^{k-1}(\theta_j, \varphi_j)\vee(\theta_k, \varphi_k)=(G\times G, A\times A)$

for each $k=2, 3, \cdots, m$.

Proof. Suppose $\mathfrak{S}$ is isomorphic to $\prod_{j=1}^{m}\mathfrak{S}_j$ under (f, g). Let (p_j, q_j): $\prod_{j=1}^{m}\mathfrak{S}_j \Rightarrow \mathfrak{S}_j$ denote the projection homomorphisms such that $p_j(x_1, \cdots, x_m)=x_j$ and $q_j(a_1, \cdots, a_m)=a_j$. For $j=1, \cdots, m$, let $h_j=p_jf$ and $k_j=q_jg$. Then, obviously $h_j(G)=G_j$ and $k_j(A)=A_j$. If $(x, x')\in\bigcap_{j=1}^{m}\theta_j$, so that $h_j(x)=p_j(f(x))=p_j(f(x'))=h_j(x')$ for $j=1, \cdots, m$, then $f(x)=f(x')$. Thus, $x=x'$ or $(x, x')\in\Delta_G$. Therefore $\bigcap_{j=1}^{m}\theta_j=\Delta_G$ and similarly $\bigcap_{j=1}^{m}\varphi_j=\Delta_A$. Let $(x, x')\in G\times G$ with $f(x)=(x_1, \cdots, x_m)$ and $f(x')=(x'_1, \cdots, x'_m)$ and let $y\in G$ be such that $f(y)=(x_1, \cdots, x_{k-1}, x'_k, \cdots)$ for any $k=1, \cdots, m$. Then $h_j(x)=p_jf(x)=p_jf(y)=h_j(y)$ for $j=1, 2, \cdots, k-1$ and $h_k(y)=p_kf(y)=p_kf(x')=h_k(x')$. In other words, $(x, y)\in\theta_j$ for $j=1, \cdots, k-1$ and hence $(x, y)\in\bigcap_{j=1}^{k-1}\theta_j$ while $(y, x')\in\theta_k$. Thus, $(x, x')\in\bigcap_{j=1}^{k-1}\theta_{j_{k-1}}\vee\theta_k$. This means $\bigcap_{j=1}^{k-1}\theta_j\vee\theta_k=G\times G$ and in a similar manner $\bigcap_{j=1}^{k-1}\varphi_j\vee\varphi_k=A\times A$ for $k=2, \cdots, m$.

The converse follows by a simple induction.

Corollary 8. *The representations of a genalgebra $\mathfrak{S}$ with permutable congruences as a direct product of a finite number of genalgebras are in one-to-one correspondence with the collection of finite sets of congruences $\{(\theta_j, \varphi_j)\,|\,j=1, \cdots, m\}$ of $\mathfrak{S}$ such that*

(1) $\bigcap_{j=1}^{m}(\theta_j, \varphi_j)=(\Delta_G, \Delta_A)$;

(2) $\bigcap_{j=1}^{k-1}(\theta_j, \varphi_j)\vee(\theta_k, \varphi_k)=(G\times G, A\times A)$ *for each* $k=2, \cdots, m$.

Theorem 9. *If the congruences of a genalgebra $\mathfrak{S}$ permute and $\mathfrak{S}$ is isomorphic to a subdirect product of simple genalgebras* $\mathfrak{S}_j=\langle G_j, o_1^j, \cdots, o_n^j, A_j\rangle (j=1, \cdots, m)$, i.e. *genalgebras with no proper congruences, then $\mathfrak{S}$ is also isomorphic with a direct product of some subset of* $\{\mathfrak{S}_j\,|\,j=1, \cdots, m\}$.

Proof. Corresponding to the simple subdirect factors $\mathfrak{S}_j$ $(j=1, \cdots, m)$ there exist reduced congruences (θ_j, φ_j) in $\mathfrak{S}$ such that $\bigcap_{j=1}^{m}(\theta_j, \varphi_j)=(\Delta_G, \Delta_A)$ and $\mathfrak{S}_j\cong\mathfrak{S}/(\theta_j, \varphi_j)$. Since $\mathfrak{S}_j$ are simple, then (θ_j, φ_j) are maximal reduced congruences. Choose a minimal subfamily of $\{\mathfrak{S}_j\,|\,j=1, \cdots, m\}$ such that its corresponding congruences satisfy

$$\bigcap_{i=1}^{r}(\theta_i, \varphi_i)=(\Delta_G, \Delta_A)\,.$$

Then, for each $k=2, \cdots, m$ note that we have

$$\bigcap_{i=1}^{k-1}(\theta_{j_i}, \varphi_{j_i})\vee(\theta_{j_k}, \varphi_{j_k})\geq(\theta_{j_k}, \varphi_{j_k}).$$

Thus, either

$$\bigcap_{i=1}^{k-1}(\theta_{j_i}, \varphi_{j_i})\vee(\theta_{j_k}, \varphi_{j_k})=(\theta_{j_k}, \varphi_{j_k})$$

or

$$\bigcap_{i=1}^{k-1}(\theta_{j_i}, \varphi_{j_i}) \vee (\theta_{j_k}, \varphi_{j_k}) = (G\times G, A\times A).$$

In the first case, then

$$\bigcap_{i=1}^{k-1}(\theta_{j_i}, \varphi_{j_i}) \le (\theta_{j_k}, \varphi_{j_k})$$

and hence by maximality,

$$\bigcap_{i=1}^{k-1}(\theta_{j_i}, \varphi_{j_i}) = (\theta_{j_k}, \varphi_{j_k})$$

contrary to the minimality of the set $\{(\theta_{j_i}, \varphi_{j_i}) \mid i=1, \cdots, r\}$. Whence the second condition prevails and the result follows.

Theorem 10. *If the lattice of congruences of a genalgebra* $\mathfrak{S}$ *is distributive and* $\mathfrak{S}$ *is isomorphic to a subdirect product of the genalgebras* $\{\mathfrak{S}_j \mid j=1, \cdots, m\}$ *under* (f, g), *then for each homomorphism* (h, k) *of* $\mathfrak{S}$ *the genalgebra* $(h,k)(\mathfrak{S})=\langle h(G), O_1, \cdots, O_n, k(A)\rangle$ *is also isomorphic to a subdirect product of homomorphic images of the genalgebras* $\mathfrak{S}_j$ $(j=1, \cdots, m)$.

Proof. By hypothesis and Theorem 1, there exists congruences (θ_j, φ_j) such that

$$\bigcap_{j=1}^{m}(\theta_j, \varphi_j) = (\varDelta_G, \varDelta_A)$$

and

$$\mathfrak{S}_j \cong \mathfrak{S}/(\theta_j, \varphi_j).$$

Let $\mathfrak{S}/(\theta, \varphi)$ be any quotient genalgebra of $\mathfrak{S}$ and hence any homomorphic image $(h, k)(\mathfrak{S})=\mathfrak{S}/(\theta, \varphi)$ of $\mathfrak{S}$. By Theorem 3, $\mathfrak{S}/(\theta, \varphi)$ is isomorphic to a subdirect product of irreducible genalgebras. Let the corresponding reduced congruences of this decomposition be given by $(\tilde{\zeta}_\mu, \tilde{\eta}_\mu)\mu \in M)$. From subdirect irreducibility, we have each of the congruences (completely) meet-irreducible. Also, each of the congruences $(\tilde{\zeta}_\mu, \tilde{\eta}_\mu)$ corresponds to a reduced congruence (ζ_μ, η_μ) of $\mathfrak{S}$. Considering an arbitrary $\mu \in M$, we have $(\zeta_\mu, \eta_\mu)=(\zeta_\mu, \eta_\mu)\vee \bigcap_{j=1}^{m}(\theta_j, \varphi_j)= \bigcap_{j=1}^{m}[(\zeta_\mu, \eta_\mu)\vee(\theta_j, \varphi_j)]$. By meet-irreduciblity, then

$$(\zeta_\mu, \eta_\mu)=(\zeta_\mu, \eta_\mu)\vee(\theta_{j_\mu}, \varphi_{j_\mu})$$

or

$$(\zeta_\mu, \eta_\mu)\ge(\theta_{j_\mu}, \varphi_{j_\mu}) \qquad \text{for some } j_\mu=1, \cdots, m\,.$$

Let then

$$(\psi_{j_\mu}, \omega_{j_\mu})=\bigcap_{(\zeta_\mu, \eta_\mu)\ge(\theta_{j_\mu}, \varphi_{j_\mu})}(\zeta_\mu, \eta_\mu).$$

If $\Omega=\{(\psi_{j_\mu}, \omega_{j_\mu}) \mid \mu \in M\}$, then

$$\bigcap_{\Theta\in\Omega}\tilde{\Theta}=(\varDelta_{G/\theta}, \varDelta_{A/\varphi})\,.$$

Thus $\mathfrak{S}/(\theta, \Psi)$ is isomorphic to a subdirect product of the genalgebras $\{(\mathfrak{S}/(\theta, \varphi))/\tilde{\Theta} \mid \Theta \in \Omega\}$ or $\{\mathfrak{S}/\Theta \mid \Theta \in \Omega\}$, and hence of

$$\{(\mathfrak{S}/\theta_{j_\mu}, \varphi_{j_\mu}))/(\tilde{\psi}_{j_\mu}, \tilde{\omega}_{j_\mu}) \mid \mu \in M\}$$

which are homomorphic images of $\mathfrak{S}_j(j=1, \cdots, m)$. Q.E.D.

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