203. Decompositions of Generalized Algebras. I

By F. M. SIOSON

Department of Mathematics, University of Florida

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In an unpublished paper [5],*) the author proposed an organic unification and generalization of the theories of G. Birkhoff's universal algebras [1], A. Tarski's relational systems [6], and G. Grätzer's multialgebras [3] (further, [2], [4]). Even under this very general setting, one is able to recapture the homomorphism theorems, the isomorphism theorems, and the Schreier-Jordan-Hölder theorems of algebra.

The unification was achieved by defining a *generalized algebra* (or simply a *genalgebra*) as a system $\mathfrak{S}=\langle G, o_1, \cdots, o_n, A\rangle$ consisting of a pair of sets G and A and a family (which may be finite or infinite) of (finitary or infinitary) functions

$$o_i : G^{m_i} \rightarrow A$$

($i=1, \cdots, n$) called operations. Thus, we have universal algebras when $A=G$; relational systems when $A=\{T, F\}$; multialgebras when $A=2^G$; and related universal algebras when $A=G\cup\{T, F\}$. The n-tuple $(m_1, \cdots, m_n)$ is called the *type* of the genalgebra. If $K\subseteq G$ and $C\subseteq A$ such that for each $i=1, \cdots, n$ and all elements $x_1, x_2, \cdots, x_{m_i}\in K$ we also have $o_i(x_1, x_2, \cdots, x_{m_i})\in C$, then $\mathcal{K}=\langle K, o_1, \cdots, o_n, C\rangle$ is said to be a *sub-genalgebra* of $\mathfrak{S}$. When C moreover is minimal, that is, when

$$C=\bigcup_{i=1}^{n} o_i(K, K, \cdots, K),$$

$\mathcal{K}$ is said to be a *reduced genalgebra*.

Given any other genalgebra $\mathcal{H}=\langle H, o'_1, \cdots, o'_n, B\rangle$ of the same type as $\mathfrak{S}=\langle G, o_1, \cdots, o_n, A\rangle$, a *homomorphism* from $\mathfrak{S}$ to $\mathcal{H}$ is a pair (h, k) of functions $h: G\rightarrow H$ and $k: A\rightarrow B$ such that for all $i=1, \cdots, n$, the following holds

$$k(o_i(x_1, x_2, \cdots, x_{m_i}))=o'_i(h(x_1), h(x_2), \cdots, h(x_{m_i}))$$

for all $x_1, x_2, \cdots, x_{m_i}\in G$. When both h and k are onto and one-to-one functions, then (h, k) is called an *isomorphism*. A *congruence* in the genalgebra $\mathfrak{S}$ is a pair (θ, φ) of equivalence relations θ on G and φ on A such that for each $i=1, \cdots, n$, if $(x_j, y_j)\in\theta$ for $j=1, 2, \cdots, m_i$, then also $(o_i(x_1, x_2, \cdots, x_{m_i}), o_i(y_1, y_2, \cdots, y_{m_i}))\in\varphi$. It should be noted that if (h, k) is a homomorphism of $\mathfrak{S}$ into $\mathcal{H}$, then (θ, φ) with $\theta=hh^{-1}$ and $\varphi=kk^{-1}$ is a congruence on $\mathfrak{S}$ (called the *kernel* of the homomorphism (h, k)). A congruence (θ, φ) on $\mathfrak{S}$ defines a new

*) For the references, see the list at the end of the following article.

genalgebra $\mathfrak{S}/(\theta, \varphi)=\langle G/\theta, \tilde{o}_1, \cdots, \tilde{o}_n, A/\varphi\rangle$ (called the *quotient genalgebra* of $\mathfrak{S}$ modulo (θ, φ)) where

$$o_i(x_1/\theta, \cdots, x_{m_i}/\theta) = o_i(x_1, \cdots, x_{m_i})/\varphi .$$

Here x_i/θ and $o_i(x_1, \cdots, x_{m_i})/\varphi$ stands for the θ-equivalence class and φ-equivalence class containing x_i and $o_i(x_1, \cdots, x_{m_i})$ respectively.

In the present communication, we shall consider direct and subdirect products and extend various theorems in algebras to genalgebras. Given a family $\{\mathfrak{S}_\lambda \mid \lambda \in \Lambda\}$ of genalgebras of the same type, where

$$\mathfrak{S}_\lambda=\langle G_\lambda, o_1^\lambda, \cdots, o_n^\lambda, A_\lambda\rangle,$$

the cartesian products $\prod_{\lambda\in\Lambda} G_\lambda$ and $\prod_{\lambda\in\Lambda} A_\lambda$ constitute a genalgebra

$$\prod_{\lambda\in\Lambda} \mathfrak{S}_\lambda=\langle \prod_{\lambda\in\Lambda} G_\lambda, O_1, \cdots, O_n, \prod_{\lambda\in\Lambda} A_\lambda\rangle$$

called the *direct product* of the genalgebras $\mathfrak{S}_\lambda$, where for $\alpha_1, \cdots, \alpha_{m_i} \in \prod_{\lambda\in\Lambda} G_\lambda$ the function $O_i(\alpha_1, \cdots, \alpha_{m_i})$ in $\prod_{\lambda\in\Lambda} A_\lambda$ is defined as follow:

$$O_i(\alpha_1, \cdots, \alpha_{m_i})(\lambda)=o_i^\lambda(\alpha_1(\lambda), \cdots, \alpha_{m_i}(\lambda))$$

for all $\lambda \in \Lambda$ and $i=1, \cdots, n$. A sub-genalgebra $\mathcal{K}=\langle K, O_1, \cdots, O_n, C\rangle$ of the direct product $\prod_{\lambda\in\Lambda} \mathfrak{S}_\lambda$ is said to be a *subdirect product* of the genalgebras $\mathfrak{S}_\lambda$ iff each of the (projection) homomorphisms (p_λ, q_λ) with $p_\lambda(\alpha)=\alpha(\lambda)$ and $q_\lambda(\beta)=\beta(\lambda)$ maps $\mathcal{K}$ *onto* $\mathfrak{S}_\lambda$, that is to say, $p_\lambda(K)=G_\lambda$ and $q_\lambda(C)=A_\lambda$.

Let us call a congruence (θ, φ) of $\mathfrak{S}$ *reduced* if φ is the smallest equivalence relation on A such that (θ, φ) is a congruence of $\mathfrak{S}$. Observe that if (θ, φ_0) is any congruence of $\mathfrak{S}$ and φ_λ is any equivalence relation on A containing φ_0, then $(\theta, \varphi_\lambda)$ is also a congruence of $\mathfrak{S}$. Thus, if $\{\varphi_\lambda \mid \lambda \in \Lambda\}$ is the collection of all equivalence relations on A such that $(\theta, \varphi_\lambda)$ is a congruence of $\mathfrak{S}$, then (θ, φ), where $\varphi=\bigcap_{\lambda\in\Lambda} \varphi_\lambda$, is a reduced congruence of $\mathfrak{S}$ (called the *reduction* of (θ, φ_0)). In many discussions of genalgebras, it is sufficient to talk only of reduced congruences.

The following results in [5] will be needed in the sequel:

Theorem A. *The collection $\mathcal{L}(\mathfrak{S})$ of all congruences in a genalgebra $\mathfrak{S}=\langle G, o_1, \cdots, o_n, A\rangle$ forms a complete, compactly generated sublattice of the direct product $\mathcal{L}(G)\times\mathcal{L}(A)$ of the lattice $\mathcal{L}(G)$ of all equivalence relations on G and the lattice $\mathcal{L}(A)$ of all equivalence relations on A.*

Theorem B. *If (h, k) is a homomorphism of $\mathfrak{S}=\langle G, o_1, \cdots, o_n, A\rangle$ into $\mathcal{H}=\langle H, o_1', \cdots, o_n', B\rangle$, then $\mathfrak{S}/(hh^{-1}, kk^{-1})$ is isomorphic to the subgenalgebra $(h, k)(\mathfrak{S})=\langle h(G), o_1, \cdots, o_n', k(B)\rangle$ of $\mathcal{H}$. Conversely, for any congruence (θ, φ) of $\mathfrak{S}$ the function pair (p, q) defined by $p(x)=x/\theta$ and $q(y)=y/\varphi$ is a homomorphism of $\mathfrak{S}$ onto $\mathfrak{S}/(\theta, \varphi)$.*

Theorem C. *For each congruence (θ, φ) of a genalgebra $\mathfrak{S}$, there is an isomorphism f between the lattice $\mathcal{L}(\mathfrak{S}/(\theta, \varphi))$ of all congruences of $\mathfrak{S}/(\theta, \varphi)$ and the lattice $\mathcal{L}(\mathfrak{S}(\theta, \varphi))$ of all congruences (ζ, η) of $\mathfrak{S}$ containing (θ, φ). Moreover, if $f(\zeta, \eta)=(\rho, \sigma)$, then*

$$\mathfrak{S}/(\zeta, \eta)\cong(\mathfrak{S}/(\theta, \varphi))/(\rho, \sigma).$$

The above function f is, in fact, defined by

$$(x, y)\in\zeta \text{ if and only if } (x/\theta, y/\theta)\in\rho,$$
$$(z, u)\in\eta \text{ if and only if } (z/\varphi, u/\varphi)\in\sigma.$$

Thus, (ρ, σ) is reduced if and only if (ζ, η) is reduced.

We now state our first result:

Theorem 1. *A necessary and sufficient condition for a genalgebra $\mathfrak{S}=\langle G, o_1, \cdots, o_n, A\rangle$ to be isomorphic with a subdirect product of the genalgebras $\mathfrak{S}_\lambda=\langle G_\lambda, o_1^\lambda, \cdots, o_n^\lambda, A_\lambda\rangle$ is that there exists for each $\lambda\in\Lambda$ a homomorphism (h_λ, k_λ) of $\mathfrak{S}$ onto $\mathfrak{S}_\lambda$ whose kernels $(\theta_\lambda, \varphi_\lambda)=(h_\lambda h_\lambda^{-1}, k_\lambda k_\lambda^{-1})$ satisfy the condition*

$$\bigcap_{\lambda\in\Lambda}(\theta_\lambda, \varphi_\lambda)=(\Delta_G, \Delta_A),$$

where $\Delta_G(\Delta_A)$ denotes the equality relation on $G(A)$.

Proof. Suppose $\mathfrak{S}$ is isomorphic under (f, g) onto the subdirect product $\mathcal{K}=\langle K, O_1, \cdots, O_n, C\rangle$ of the genalgebras $\mathfrak{S}_\lambda$. Let (p_λ, q_λ) be the projection homomorphisms of $\mathcal{K}$ onto $\mathfrak{S}_\lambda(\lambda\in\Lambda)$. Set $(h_\lambda, k_\lambda)=(p_\lambda f, q_\lambda g)$. Then for each $i=1, \cdots, n$ and all $x_1, \cdots, x_{m_i}\in G$, we have

$$\begin{aligned}k_\lambda(o_i(x_1, \cdots, x_{m_i}))&=q_\lambda g(o_i(x_1, \cdots, x_{m_i}))=q_\lambda O_i(f(x_1), \cdots, f(x_{m_i}))\\&=o_i^\lambda(p_\lambda f(x_1), \cdots, p_\lambda f(x_{m_i}))=o_i^\lambda(h_\lambda(x_1), \cdots, h_\lambda(x_{m_i})).\end{aligned}$$

Thus, (h_λ, k_λ) is a homomorphism, and clearly $h_\lambda(G)=G_\lambda$ and $k_\lambda(A)=A_\lambda$. It remains to show the second condition. Assume that $(x, y)\in\bigcap_{\lambda\in\Lambda}\theta_\lambda$ so that $(x, y)\in\theta_\lambda$ for all $\lambda\in\Lambda$. Since $\theta_\lambda=h_\lambda h_\lambda^{-1}=(p_\lambda f)(p_\lambda f)^{-1}$, then $p_\lambda(f(x))=p_\lambda(f(y))$ for all $\lambda\in\Lambda$. Hence $f(x)=f(y)$. But f is one-to-one; therefore, $x=y$ or $(x, y)\in\Delta_G$. A similar result may be derived for Δ_A. The second given condition thus follows.

Conversely, suppose the above two conditions hold. For each $x\in G$ and $y\in A$, let $\chi\in\prod_{\lambda\in\Lambda}G_\lambda$ and $\psi\in\prod_{\lambda\in\Lambda}A_\lambda$ such that $\chi(\lambda)=h_\lambda(x)$ and $\psi(\lambda)=k_\lambda(y)$ for all $\lambda\in\Lambda$. Set $K=\{\chi\mid x\in G\}$ and $C=\{\psi\mid y\in A\}$. For each $i=1, 2, \cdots, n$ and $\chi_1, \cdots, \chi_{m_i}\in K$ observe that

$$\begin{aligned}O_i(\chi_1, \cdots, \chi_{m_i})(\lambda)&=o_i^\lambda(\chi_1(\lambda), \cdots, \chi_{m_i}(\lambda))=o_i^\lambda(h_\lambda(x_1), \cdots, h_\lambda(x_{m_i}))\\&=k_\lambda(o_i(x_1, \cdots, x_{m_i}))\in C.\end{aligned}$$

Hence $\mathcal{K}=\langle K, O_1, \cdots, O_n, C\rangle$ is a subgenalgebra of the product $\prod_{\lambda\in\Lambda}\mathfrak{S}_\lambda$.

Define $(f, g)\colon\mathfrak{S}\to\mathcal{K}$ such that $f(x)=\chi$ and $g(y)=\psi$. Let

$$g(o_i(x_1, \cdots, x_{m_i}))=\omega$$

for $i=1, \cdots, n$ and $x_1, \cdots, x_{m_i}\in G$. Then

$$\begin{aligned}\omega(\lambda)&=k_\lambda(o_i(x_1, \cdots, x_{m_i}))=o_i^\lambda(h_\lambda(x_1), \cdots, h_\lambda(x_{m_i}))\\&=o_i^\lambda(\chi_1(\lambda), \cdots, \chi_{m_i}(\lambda))=O_i(\chi_1, \cdots, \chi_{m_i})(\lambda)\end{aligned}$$

for all $\lambda \in \Lambda$ and therefore $O_i(\chi_1, \cdots, \chi_{m_i}) = \omega$. Hence

$$g(o_i(X_1, \cdots, x_{m_i})) = \omega = O_i(\chi_1, \cdots, \chi_{m_i}) = O_i(f(x_1), \cdots, f(x_{m_i})).$$

This shows that (f, g) is a homomorphism. If $f(x_1) = f(x_2)$ so that $\chi_1(\lambda) = \chi_2(\lambda)$ for all $\lambda \in \Lambda$, then $h_\lambda(x_1) = h_\lambda(x_2)$ for all $\lambda \in \Lambda$. This means that $(x_1, x_2) \in \theta_\lambda$ for all $\lambda \in \Lambda$ and therefore by hypothesis $x_1 = x_2$. The same proof applies to the function g. Thus, both f and g are one-to-one and therefore (f, g) is an isomorphism. Inasmuch as $h_\lambda(G) = G_\lambda$ and $k_\lambda(A) = A_\lambda$ for each $\lambda \in \Lambda$, then $p_\lambda(K) = h_\lambda f^{-1}(K) = h_\lambda(G) = G_\lambda$ and $q_\lambda(C) = k_\lambda g^{-1}(C) = k_\lambda(A) = A_\lambda$. This completes the proof of the theorem.

Corollary 2. *There is a one-to-one correspondence between the subdirect product representations of a genalgebra $\mathfrak{S}$ and the collection of all sets of congruences $\{(\theta_\lambda, \varphi_\lambda) \mid \lambda \in \Lambda\}$ of $\mathfrak{S}$ such that*

$$\bigcap_{\lambda \in \Lambda} (\theta_\lambda, \varphi_\lambda) = (\Delta_G, \Delta_A).$$

A genalgebra is said to be *directly* (*subdirectly*) *reducible* iff it is isomorphic to a direct (subdirect) product of at least two genalgebras such that of the associated projection homomorphisms are isomorphisms (and hence with non-trivial projection kernels). Otherwise, it is *directly* (*subdirectly*) *irreducible.*

Examples. Consider the genalgebra $\mathfrak{S} = \langle G, o, A \rangle = \langle \{0, 1\}, o, \{a, b\} \rangle$ such that $o(x, y) = a$ for all $x, y \in G$. Then note that $\mathfrak{S} \cong \mathfrak{S}/(G \times G, \Delta_A) \times \mathfrak{S}/(\Delta_G, A \times A)$ under the function pair defined by

$$f(0) = (\{0, 1\}, \{0\}) \quad \text{and} \quad g(a) = (\{a\}, \{a, b\})$$
$$f(1) = (\{0, 1\}, \{1\}) \qquad\qquad g(b) = (\{b\}, \{a, b\}).$$

Note also that $\mathfrak{S}$ is isomorphic to the subdirect product

$$\langle \{(\{0\}, \{0, 1\}), (\{1\}, \{0, 1\})\}, O, \{(\{a\}, \{a\}), (\{b\}, \{b\})\} \rangle$$

of $\mathfrak{S}/(\Delta_G, \Delta_A)$ and $\mathfrak{S}/(G \times G, \Delta_A)$ under the function pair (f, g) such that

$$f(0) = (\{0\}, \{0, 1\}) \quad \text{and} \quad g(a) = (\{a\}, \{a\})$$
$$f(1) = (\{1\}, \{0, 1\}) \qquad\qquad g(b) = (\{b\}, \{b\}).$$

On the other hand, observe that the reduced genalgebra $\mathfrak{S}' = \langle \{0, 1\}, o, \{a\} \rangle$ is both subdirectly and directly irreducible.