202. Connection of Topological Vector Bundles

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In this note, we show the notions of connection and curvature can be defined for arbitrary vector bundles (not necessarily the dimension of fibre be finite) over paracompact normal topological space. In usual differential geometry, Nomizu's theorem (Nomizu [4], [5]) and Ambrose-Singer's theorem (Ambrose-Singer [1], Nomizu [5]) show that the connected component of the structure group of a vector boundle is determined by its curvature form. This is also true in our case. (Theorem [2]).

1. **Alexander-Spanier cohomology.** Let X be a paracompact normal topological space. We set

$$\Delta_s(X) = \{(x, x, \cdots, x)\} \subset \overbrace{X \times X \times \cdots \times X}^{s+1}.$$

By definition, $\Delta_0(X) = X$.

We denote by L a topological vector space over K, where K is R or C. f is an L-valued continuous function on some neighborhood of $\Delta_s(X)$ in $X \times \cdots \times X$ such that

$$f(x_0, \cdots, x_{s+1}) = 0, \quad \text{if } x_i = x_{i+1} \text{ for some } i, \quad 0 \leq i \leq s.$$

If f and f' are two such functions, then we call f and f' are equivalent if

$$f \mid U(\Delta_s(X)) = f' \mid U(\Delta_s(X)),$$

where $U(\Delta_s(X))$ is a neighborhood of $\Delta_s(X)$. The set of all these equivalence classes is denoted by $C^s(X, L)$ or $C^s(X)$. For simplicity, the class of f is also denoted by f.

We define the homomorphism $d: C^s(X) \to C^{s+1}(X)$ by

$$(df)(x_0, \cdots, x_{s+1}) = \sum_{i=0}^{s+1}(-1)^i f(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{s+1}).$$

Then $d^2 = 0$. Moreover, setting

$$(k_a f)(x_0, \cdots, x_{s-1}) = f(a, x_0, \cdots, x_{s-1}),$$

$k_a^2 = 0$, and we obtain $(dk_a + k_a d)f = f$ if $\deg. f \geq 1$, and $(dk_a + k_a d)f = f - f(a)$ if $\deg. f = 0$, locally. If we denote ker. $[d: C^s(X) \to C^{s+1}(X)] = F^s(X)$, then we know

$$H^s(X, L) \simeq F^s(X)/dC^{s-1}(X) \quad s \geq 0,$$

where the left hand side is Čech cohomology group. (Godement [3]).

2. **The s-cross-section.** We fix a fibre bundle ξ with base space X, fibre L and structure group G.

We assume that G satisfies following 3-conditions.

(i) The element of G are linear transformations of L.

(ii) There is a topological ring A over K of linear operators of L, which contains G as a topological subgroup and the map $A \times L \to L((a, x) \to ax)$ is continuous under the topology of A.

(iii) There is a neighborhood $U(1)$ of the identity of A such that $U(1) \subset G$.

For example, taking $A = \mathfrak{gl}(n, K)$, $G = GL(n, K)$ satisfies these conditions.

We denote the coordinate neighborhood system of ξ by $\{U\}$ and the transition functions by $\{\phi_{UV}\}$. We assume $\{U\}$ is locally finite.

Definition. If the collection $\{f_U\}$, $f_U \in C^s(U)$ satisfies

(1) $\phi_{UV}(x_0) f_V(x_0, \cdots, x_s) = f_U(x_0, \cdots, x_s)$, $x_0 \in U \cap V$,

then we call $\{f_U\}$ is an s-cross-section of ξ.

By definition, the 0-cross-section is the usual cross-section of ξ. The set of all s-cross-sections of ξ form a K-vector space, which we denote by $C^s(X, \xi)$.

3. *Definition of connection forms.* As A is a topological K-vector space, we can define $C^s(X, A)$. $C(X, A) = \sum_{s \geq 0} C^s(X, A)$ is a graded ring by the multiplication

$$(\varphi \cdot \psi)(x_0, \cdots, x_{s+t}) = \varphi(x_0, \cdots, x_s) \psi(x_s, \cdots, x_{s+t}).$$

Moreover, we can define the operation of $\varphi \in C^s(X, A)$ to $f \in C^t(X)$ by

$$(\varphi f)(x_0, \cdots, x_{s+t}) = \varphi(x_0, \cdots, x_s) f(x_s, \cdots, x_{s+t}).$$

By definition, φf belongs in $C^{s+t}(X)$.

On $C(X, A)$, we can define d and locally k_a, and we get

(2) $d(\varphi \cdot \psi) = d\varphi \cdot \psi + (-1)^s \varphi d\psi$, $\varphi \in C^s(X, A)$, $\psi \in C^t(X, A)$ or $C^t(X)$.

$k_a(\varphi \cdot \psi) = (k_a \varphi) \psi$. deg. $\varphi \geq 1$, $k_a(\varphi \cdot \psi) = \varphi(a) \cdot k_a \psi$, deg. $\varphi = 0$.

(3) $(dk_a + k_a d)\varphi = \varphi$, $\varphi \in C^s(U(a), A)$, $s \geq 1$,

$(dk_a + k_a d)\varphi = k_a d\varphi = \varphi - \varphi(a)$, $\varphi \in C^0(U(a), A)$,

where $U(a)$ is a neighborhood of a.

Definition. A collection $\{\theta_U\}$, $\theta_U \in C^1(U, A)$, is called a connection form of ξ if $\{\theta_U\}$ satisfies

$(d + \theta_U) f_U = \phi_{UV}(d + \theta_V) f_V$, for all $\{f_U\} \in C^s(X, \xi)$, $s \geq 0$.

Note. In this definition, connection form depends on transition $\{\phi_{UV}\}$ and covering $\{U\}$. If $\{\theta_U\}$ and $\{\theta_{U'}\}$ are connection forms of $(\{\phi_{UV}\}, \{U\})$ and $(\{\phi_{U'V'}\}, \{U'\})$ such that there exists a common refinement $\{U''\}$ of $\{U\}$ and $\{U'\}$ and $\phi_{UV} \mid U'' \cap V'' = \phi_{U'V'} \mid U'' \cap V''$, $\theta_U \mid U'' = \theta_{U'} = \mid U''$, then we identify $\{\theta_U\}$ and $\{\theta_{U'}\}$. The relation between the change of transition functions and connection forms is stated in n°5, (iii).

Definition. For $\{f_U\} \in C^s(X, \xi)$, we set

$$D\{f_U\} = \{(d + \theta_U) f_U),$$

and call D is the covariant derivation.

By definition, D is a homomorphism from $C^s(X, \xi)$ to $C^{s+1}(X, \xi)$.

Definition. $\{\Theta_U\} = \{d\theta_U + \theta_U\theta_U\}$ is called the curvature form of $\{\theta_U\}$.

4. Existence of connection forms. **Lemma 1.** (Cf. Chern [2], § 12). $\{\theta_U\}$, $\theta_U \in C^1(U, A)$ becomes a connection form of ξ if and only if they satisfy

(4) $$\theta_U\phi_{UV} + d\phi_{UV} = \phi_{UV}\theta_V.$$

Theorem 1. Connection form always exists.

Proof. We set
$$\theta_U(x_0, x_1) = \sum_{U \cap W \neq \phi} e_W(x_0)\phi_{UW}(x_0)(d\phi_{WU})(x_0, x_1)$$
$$= \sum_{U \cap W \neq \phi} e_W(x_0)\phi_{UW}(x_0)(\phi_{WU}(x_1) - \phi_{WU}(x_0)),$$

where $\{e_U(x)\}$ is the partition of unity corresponding to $\{U\}$.

By definition, $\theta_U \in C^1(X, A)$ and $\theta_U(x, x) = 0$. Moreover, we get on $U \cap V$

$$\theta_U(x_0, x_1)\phi_{UV}(x_1) + \phi_{UV}(x_1) - \phi_{UV}(x_0)$$
$$= \sum_{U \cap V \cap W \neq \phi} e_W(x_0)(\phi_{UW}(x_0)\phi_{WV}(x_1) - \phi_{UV}(x_1)) + \phi_{UV}(x_1) - \phi_{UV}(x_0)$$
$$= \phi_{UV}(x_0)(\sum_{U \cap V \cap W \neq \phi} e_W(x_0)(\phi_{WV}(x_1) - \phi_{WV}(x_0)))$$
$$= \phi_{UV}(x_0)\theta_V(x_0, x_1).$$

Hence $\{\theta_U\}$ is a connection form of ξ.

5. Some formulas. In this n^o, we summarize formulas of connection forms and curvature forms. Except (6), these are known in usual differential geometry. Since the proofs of these formulas are similar that of in usual differential geometry, we omitt them. (Cf. Chern [2], § 12, Nomizu [5]).

(i) $\{\theta_U'\}$, $\theta_U' \in C^1(U, A)$ becomes a connection form of $\xi = \{\phi_{UV}\}$, if and only if, setting $\theta_U' = \theta_U + \eta_U$, $\{\eta_U\}$ satisfies $\eta_U(x, x) = 0$ and $\phi_{UV}^{-1}\eta_U\phi_{UV} = \eta_V$. In this case, the curvature form Θ_U' is given by

(5) $$\Theta_U' = \Theta_U + d\eta_U + \theta_U\eta_U + \eta_U\theta_U + \eta_U\eta_U.$$

(ii) $\{\Theta_U\}$ satisfies $\phi_{UV}^{-1}\Theta_U\phi_{UV} = \Theta_V$ and since $\Theta_U \in C^2(U, A)$,

(6) $$k_a^2\Theta_U = 0.$$

(iii) If $\{\theta_U\}$ is a connection form of $\{\phi_{UV}\}$ then $\{h_U(\theta_U - h_U^{-1}dh_U)h_U^{-1}\}$ becomes a connection form of $\{h_U\phi_{UV}h_V^{-1}\}$ and its curvature form Θ_U' is given by

(5)' $$\Theta_U' = h_U\Theta_U h_U^{-1},$$

where h_U is a continuous function from U into G.

(iv) $D\Theta_U = \Theta_U\eta_U$. In general, we obtain $D^2f_U = \Theta_U f_U$.

6. Reduction theorem. **Lemma 2.** Setting $h_{U,a} = (1 + k_a\theta_U)$, we obtain

(7) $$dh_{U,a} = h_{U,a}\theta_U - k_a\Theta_U.$$

Proof. As $dh_{U,a} = dk_a\theta_U = \theta_U - k_ad\theta_U$ and $d\theta_U = \Theta_U - \theta_U\theta_U$, we have

(7).

For simplicity, we denote h_U instead of $h_{U,a}$.

Lemma 3. There exists a locally finite covering $\{U\}$ of X and point set $\{a_U \mid a_U \in U\}$ such that for any $U \in \{U\}$, θ_U is defined on $U \times U$ and (2), (3), (4) are true on U. Moreover, $h_{U,a_U}(x_0)$ is defined and belongs in G for all $x_0 \in U$.

In next theorem, we denote $a_U = a$, $a_V = b$.

Theorem 2. We assume $\{U\}$ satisfies the statement of lemma 3. Then we obtain

$$(8) \qquad (1 + k_a k_b \theta'_V)(x_0) = (h_U \phi_{UV} h_V^{-1})(a)^{-1} (h_U \phi_{UV} h_V^{-1})(x_0),$$
$$x_0 \in U \cap V, \quad \theta'_V = h_V \theta_V h_V^{-1}.$$

Proof. By (7), we get $h_U^{-1} dh_U = \theta_U - h_U^{-1} k_a \theta_U$. Hence we have by (4)

$$d(h_U \phi_{UV} h_V^{-1})$$
$$= dh_U \phi_{UV} h_V^{-1} + h_U d\phi_{UV} h_V^{-1} - h_U \phi_{UV} h_V^{-1} dh_V h_V^{-1}$$
$$= h_U (h_U^{-1} dh_U \phi_{UV} + d\phi_{UV} - \phi_{UV} h_V^{-1} dh_V) h_V^{-1}$$
$$= h_U (\theta_U \phi_{UV} + d\phi_{UV} - \phi_{UV} \theta_V + \phi_{UV} h_V^{-1} k_b \theta_V - h_U^{-1} k_a \theta_U \phi_{UV}) h_V^{-1}$$
$$= h_U (\phi_{UV} h_V^{-1} k_b \theta_V - h_U^{-1} k_a \theta_U \phi_{UV}) h_V^{-1}.$$

Then by (3) and (6), we obtain

$$h_U \phi_{UV} h_V^{-1}(x_0) - h_U \phi_{UV} h_V^{-1}(a) = (h_U \phi_{UV} h_V^{-1})(a) k_a k_b \theta_V h_V^{-1}.$$

On the other hand, we know that $k_b h_V = k_b (1 + k_b \theta_V) = 1$. Therefore we have (8).

Corollary 1. (Cf. Ambrose-Singer [1], Nomizu [4], [5]). If $(1 + k_a k_b \theta'_V)(x_0) \in H$ for all a, b, and x_0, where H is a subgroup of G, then the connected component of the structure group of ξ is reduced to a subgroup of H.

Corollary 2. If ξ allows a connection form with curvature form 0, then the associated principal bundle of ξ is induced from a representation of $\pi_1(X)$ in G.

This follows from corollary 1 and Steenrod [6], § 13.

References

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