199. *Axiom Systems of B-algebra. II*

By Yoshinari ARAI and Kiyoshi ISÉKI

(Comm. by Kinjirô KUNUGI, M.J.A., Dec. 13, 1965)

In the first note [2], we gave axiom systems of B-algebra. A B-algebra $M=\langle x, 0, *, \sim\rangle$ is given by the following axioms:

B 1 $x*y\leqslant x$,

B 2 $(x*z)*(y*z)\leqslant(x*y)*z$,

B 3 $x*y\leqslant(\sim y)*(\sim x)$,

B 4 $0\leqslant x$,

where $x\leqslant y$ means $x*y=0$, and if $x\leqslant y$, $y\leqslant x$, then we write $x=y$. There are some axiom systems which is equivalent to $B\,1\sim B\,4$. For the details, see [1],[2], and [3].

In this note, we shall show the following

Theorem. *A B-algebra* $M=\langle X, 0, *, \sim\rangle$ *is characterized* by

L 1 $x*(\sim y)\leqslant x*(z*y)$,

L 2 $x*y\leqslant x*(y*z)$,

L 3 $(x*(y*z))*(x*y)\leqslant x*(\sim z)$,

L 4 $0\leqslant x$.

The conditions $L\,1\sim L\,4$ are an algebraic formulation of Lukasiewicz axioms of classical propositional calculus.

We first prove $B\Rightarrow L$.

As shown in [1], if $x\leqslant y$ in a B-algebra, then $z*y\leqslant z*x$ for any $z\in X$. Hence, by B 1, we have $x*y\leqslant x*(y*z)$. On the other hand, by (8) in [1], $z*y\leqslant\sim y$. Therefore we have $x*(\sim y)\leqslant x*(z*y)$. Next we have the following relation.

$$(x*(y*z))*(x*y)=(\sim(y*z)*(\sim x))*(\sim y*\sim x)\leqslant(\sim(y*z)*\sim y)*(\sim x)$$
$$=(y*(y*z))*\sim x\leqslant x*\sim(y*(y*z)).$$

On the other hand, by $y*z\leqslant y*z$, we have $y*(y*z)\leqslant z$. Hence $\sim z\leqslant\sim(y*(y*z))$. Therefore we have

$$(x*(y*z))*(x*y)\leqslant x*\sim(y*(y*z))\leqslant x*(\sim z),$$

which completes the proof of $B\Rightarrow L$.

Now we shall prove $L\Rightarrow B$.

From L 1 and L 2, we have

(1) $x\leqslant y*z$ implies $x\leqslant\sim z$ and $x\leqslant y$.

By L 2, we have $(x*y)*x\leqslant(x*y)*(x*(y*z))=0$. Hence

(2) $x*y\leqslant x$,

which is B 1. From L 3, we have

(3) $x\leqslant\sim z$ implies $x*(y*z)\leqslant x*y$.

(4) $x\leqslant\sim z$, $x\leqslant y$ imply $x\leqslant y*z$.

By L 1 and (2), we have

$$(((x*x)*y)*\sim x)\leqslant((x*x)*y)*(x*x)=0,$$

hence

(5) $$(x*x)*y\leqslant\sim x.$$

By (4), (5), and $(x*x)*y\leqslant x*x$, we have $(x*x)*y\leqslant(x*x)*x=0$, and y is arbitrary, hence

(6) $$x*x=0, \text{ i.e. } x\leqslant x.$$

Put $x=x*y, z=x$ in (1), then we have $(x*y)*\sim y\leqslant(x*y)*(x*y)=0$, hence

(7) $$x*y\leqslant\sim y.$$

In $L\,2$, put $x=(x*z)*(y*z)$, $y=x$, then we have

$$((x*z)*(y*z))*x\leqslant((x*z)*(y*z))*(x*z)=0,$$

by (2). Hence

(8) $$(x*z)*(y*z)\leqslant x.$$

$x*z\leqslant\sim z$ and (3) imply

$$((x*z)*(y*z))*((x*z)*y)\leqslant(x*z)*(\sim z)=0,$$

hence we have

(9) $$(x*z)*(y*z)\leqslant(x*z)*y.$$

Put $x=(x*z)*(y*z)$, $z=x*z$ in $L\,1$, then by (9), we have

$$(((x*z)*(y*z))*\sim y)\leqslant((x*z)*(y*z))*((x*z)*y)=0,$$

hence

(10) $$(x*z)*(y*z)\leqslant\sim y.$$

By $L\,1$ and (2), we have

$$((x*z)*(y*z))*\sim z\leqslant((x*z)*(y*z))*(x*z)=0,$$

hence

(11) $$(x*z)*(y*z)\leqslant\sim z.$$

Next we shall prove $B\,2$. By (8) and (10), if we use (4), then we have

(12) $$(x*z)*(y*z)\leqslant x*y,$$

and further by (11), (12), we have

(13) $$(x*z)*(y*z)\leqslant(x*y)*z,$$

which is axiom $B\,2$.

Moreover we must prove axiom $B\,3$.

By $L\,2$ and (2), we have

$$((x*y)*z)*x\leqslant((x*y)*z)*(x*y)=0.$$

By $L\,1$ and (2), we have

$$((x*y)*z)*\sim y\leqslant((x*y)*z)*(x*y)=0,$$

and

$$((x*y)*z)*\sim z\leqslant((x*y)*z)*((x*y)*z)=0.$$

Therefore we have

(a) $$(x*y)*z\leqslant x,$$

(b) $$(x*y)*z\leqslant\sim y,$$

(c) $$(x*y)*z\leqslant\sim z,$$

hence, using (4), we have

$$(x*y)*z \leqslant x*z.$$

This formula and (b) imply

(14) $$(x*y)*z \leqslant (x*z)*y.$$

Formula (14) implies a commutative law:

(15) If $x*y \leqslant z$, then $x*z \leqslant y$.

Substitute $x = x*(x*x)$, $y = x*(\sim x)$, and $z = x*x$ in (15), then

$$((x*(x*x))*(x*\sim x))*(x*x) \leqslant (((x*(x*x))*(x*x))*(x*\sim x).$$

The right side is equal to 0 by $L\,3$, hence

(16) $$x*(x*x) \leqslant x*\sim x.$$

By the commutative law,

(17) $$x*(x*\sim x)=0.$$

From (14), we have $(x*z)*(x*y) \leqslant y*z$ by the commutative law. In the formula, put $y = x*\sim x$, $z = \sim(\sim x)$ then

$$(x*\sim(\sim x))*(x*(x*(\sim x))) \leqslant (x*\sim x)*\sim(\sim x).$$

By (7), the right side is 0, and by (17), the second term of the left side is 0, hence

(18) $$x \leqslant \sim(\sim x).$$

(14) means that $x \leqslant y$, $y \leqslant z$ imply $x \leqslant z$. Therefore $x*y \leqslant x$ and $x \leqslant \sim(\sim x)$ imply $x*y \leqslant \sim(\sim x)$. On the other hand, $x*y \leqslant \sim y$ by (7). Formula (4) implies

$$x*y \leqslant (\sim y)*(\sim x),$$

which is axiom $B\,3$.

Therefore we complete the proof of Theorem.

References

[1] K. Iséki: Algebraic formulations of propositional calculi. Proc. Japan Acad., **41**, 803-807 (1965).

[2] ——: Axiom systems of B-algebra. Proc. Japan Acad., **41**, 808-811 (1965).

[3] ——: A characterization of Boolean algebra. Proc. Japan Acad., **41**, 893-897 (1965).