195. A Characterization of Boolean Algebra

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In my note [4], I gave an algebraic formulation of propositional calculi. Under the same idea, I shall give a characterization of Boolean algebra. The fundamental axioms are given by the algebraic formulation of E. Mendelson axioms [3].

Let $\langle X, 0, *, \sim\rangle$ be an algebra satisfying the axioms given below, where 0 is an element of a set X, and $*$ is a binary operation and $\sim$ is an unary operation. If $x*y=0$, $x, y\in X$, we write $x\leqslant y$ and $\leqslant$ introduce an order relation on X.

1 $x*y\leqslant x$,

2 $(x*z)*(y*z)\leqslant(x*y)*z$,

3 $x*(y*\sim x)\leqslant(\sim y)*(\sim x)$,

4 $0\leqslant x$,

5 $x\leqslant y$ and $y\leqslant x$ imply $x=y$.

If we define $x\vee y=\sim(\sim y*x)$, $x\wedge y=y*(\sim x)$ *and put* $1=\sim 0$, *then* we shall prove that *the algebra* $M=\langle X, 0, *, \sim\rangle$ *is a Boolean algebra with* 1 *as the unit on the operations* $\vee$, $\wedge$, *and* $\sim$. To prove it, we need some lemmas given in [4]. We do not give the proofs of these lemmas (see [4]).

(1) $0*x=0$.

(2) $x*x=0$.

(3) $x\leqslant y$ and $y\leqslant z$ imply $x\leqslant z$.

(4) $x*y\leqslant z$ implies $x*z\leqslant y$.

(5) $x\leqslant y$ implies $z*y\leqslant z*x$ and $x*z\leqslant y*z$.

(6) $y*x=(y*x)*x$.

The relation of (4) is called the *commutative law*. Further we shall prove some propositions from the axioms and the propositions (1)$\sim$(6) which are proved from the axioms 1 and 2.

(7) $x*(x*(\sim x))=0$, $x*(\sim x)=x$.

Let $y=x$ in axiom 3, then by axiom 2, we have (7) and $x*(\sim x)=x$ by axiom 1.

(8) $x*(\sim y)\leqslant y*(\sim x)$.

From axioms 1 and 3, we have

$$x*(y*(\sim x))\leqslant(\sim y)*(\sim x)\leqslant\sim y,$$

hence by (4), we have $x*(\sim y)\leqslant y*(\sim x)$.

(9) $x*(\sim y)=y*(\sim x)$.

This follows from (8) (see [4]), and consequently M is a BN-algebra. Hence we have $x\leqslant\sim(\sim x)$, and M is an NBN-algebra by theorem 4 in [4]. Hence for any $x, y \in X$, we have $(\sim x)*(\sim y) \leqslant y*x$.

(10) $x*(\sim y) \leqslant y, \quad x*y \leqslant \sim y$.

By (9) and axiom 1, we have $x*(\sim y) = y*(\sim x) \leqslant y$. Next, if we apply the commutative law, then we have $x*y \leqslant (\sim y)$.

(11) $z*y \leqslant z*(x*(\sim y))$.

The first inequality of (10) and (5) imply (11).

(12) $z*(\sim x) \leqslant z*(y*x)$.

This follows from $y*x \leqslant \sim x$ and (5).

As already noted, $\boldsymbol{M}$ is an NBN-algebra, so $(\sim x)*(\sim y) \leqslant y*x$ holds in $\boldsymbol{M}$. Hence by (5), we have

$$(y*(x*(\sim y)))*(y*x) \leqslant (y*(x*(\sim y)))*((\sim x)*(\sim y)).$$

By axiom 3, the right side of inequality above is 0, hence we have

$$y*(x*(\sim y)) \leqslant y*x.$$

Further, by the commutative law, we have

(13) $y*(y*x) \leqslant x*(\sim y)$.

Put $y = \sim x$ in (13), then we have $(\sim x)*((\sim x)*x) \leqslant x*(\sim(\sim x)) = 0$ by $x \leqslant \sim(\sim x)$. Hence

(14) $(\sim x)*((\sim x)*x) = 0$.

Let $z = \sim(\sim x)$, $x = \sim(\sim x)$, and $y = x$ in (11), then by (14), we have

(15) $\sim(\sim x) \leqslant x$.

On the other hand, $x \leqslant \sim(\sim x)$ holds in $\boldsymbol{M}$, hence

(16) $\sim(\sim x) = x$.

In the BN-axiom (9) $x*(\sim y) = y*(\sim x)$, substitute $\sim x$ for x, then, by (16), we have $(\sim x)*(\sim y) = y*(\sim(\sim x)) = y*x$. Therefore the B-axiom holds, and the NB-axiom also holds in $\boldsymbol{M}$.

In (13), if we substitute x for y, and 0 for x, then we have $x*(x*0) \leqslant 0*(\sim x) = 0$. Hence $x \leqslant x*0$. Axiom 1 implies $x*0 \leqslant x$, therefore we have

(17) $x*0 = x$.

We first verify some simple axioms of the Boolean algebra (for axioms, see [1] or [2]). Several axioms verified are superfluous.

(A) $\sim(\sim x) = x$

is proposition (16).

(B) $x \vee y = y \vee x, \quad x \wedge y = y \wedge x$

are proved by $x \vee y = \sim(\sim y*x) = \sim(\sim x*y) = y \vee x$ and $x \wedge y = y*(\sim x) = x*(\sim y) = y \wedge x$.

(C) $x \wedge 0 = 0, \quad x \vee 1 = 1$

follow from $x \wedge 0 = 0*(\sim x) = 0$ and $x \vee 1 = \sim(\sim 1*x) = \sim(0*x) = \sim 0 = 1$.

(D) $x \wedge 1 = x, \quad x \vee 0 = x$

are proved from $x \wedge 1 = 1*(\sim x) = x*0 = x$ and $x \vee 0 = \sim(\sim 0*x) = \sim((\sim x)*0) = \sim(\sim x) = x$ by using (16) and (17).

(E) $x \wedge (\sim x) = 0,\ x \vee (\sim x) = 1$

follow from $x \wedge (\sim x) = \sim x * \sim x = 0$ and $x \vee (\sim x) = \sim(\sim(\sim x) * x) = \sim(x * x) = \sim 0 = 1$.

(F) $x \wedge x = x,\ x \vee x = x$

are proved by $x \wedge x = x * (\sim x) = x$, and $x \vee x = \sim(\sim x * x) = \sim(\sim x) = x$ by using (17).

(G) $\sim(x \wedge y) = \sim x \vee \sim y,\ \sim(x \vee y) = \sim x \wedge \sim y$

follow from $\sim(x \wedge y) = \sim(y * (\sim x)) = \sim(\sim(\sim x) * (\sim y)) = (\sim y) \vee (\sim x) = (\sim x) \vee (\sim y)$, and $\sim(x \vee y) = \sim(\sim((\sim y) * x)) = (\sim y) * x = (\sim y) * (\sim(\sim x)) = (\sim x) \wedge (\sim y)$.

Next we shall prove $x \wedge (x \vee y) = x,\ x \vee (x \wedge y) = x$. We first prove

(18) $x * (x * (\sim y * x)) = 0$, i.e. $x * (\sim y * x) = x$.

By (13) and the NB-formula, we have

$$x * (x * (\sim y * x)) \leqslant (\sim y * x) * (\sim x) = (\sim x * y) * (\sim x).$$

By axiom 1 and (15), $(\sim x * y) * (\sim x) \leqslant (\sim x) * (\sim x) = 0$. Therefore we have the formula (18).

Consider $x \wedge (x \vee y)$, then by the definitions of $\wedge$, $\vee$, we have $x \wedge (x \vee y) = x \wedge (\sim(\sim y * x)) = \sim(\sim y * x) * (\sim x) = x * ((\sim y) * x) = x$ from (18). By the same technique, we have $x \vee (x \wedge y) = x \vee (y * (\sim x)) = \sim(\sim(y * (\sim x)) * x) = \sim(\sim x * (y * (\sim x))) = \sim(\sim x) = x$. Hence

(H) $x \wedge (x \vee y) = x,\ x \vee (x \wedge y) = x$.

Now we are a position to prove the associative law on $\wedge$, $\vee$, and distributive law. If we prove these properties, $\boldsymbol{M}$ is a Boolean algebra (see [2]). We shall consider the proof of

(I) $x \wedge (y \wedge z) = (x \wedge y) \wedge z,\ x \vee (y \vee z) = (x \vee y) \vee z$.

By the definitions of $\wedge$ and $\vee$, we have

$$x \wedge (y \wedge z) = (y \wedge z) * (\sim x) = (z * (\sim y)) * (\sim x) = (y * (\sim z)) * (\sim x),$$
$$(x \wedge y) \wedge z = z * (\sim(x \wedge y)) = z * (\sim(y * (\sim x))) = (y * (\sim x)) * (\sim z)$$

and

$$x \vee (y \vee z) = x \vee (\sim(\sim z * y)) = \sim((\sim z * y) * x) = \sim((\sim y * z) * x),$$
$$(x \vee y) \vee z = (\sim(\sim y * z)) \wedge z = \sim(\sim z * (\sim(\sim y * x))) = \sim((\sim y * x) * z).$$

To prove (I), it is sufficient to verify

(19) $(x * y) * z = (x * z) * y$.

By axioms 2 and 1, we have

$$(x * y) * ((x * z) * y) \leqslant (x * (x * z)) * y \leqslant x * (x * z).$$

From (13) and axiom 1, we have

$$x * (x * z) \leqslant z * (\sim x) \leqslant z.$$

Hence $(x * y) * ((x * z) * y) \leqslant z$. Applying the commutative law, we have

$$(x * y) * z \leqslant (x * z) * y.$$

This implies the formula (19).

Therefore, we have that $\boldsymbol{M}$ is a lattice and $x \vee y, x \wedge y$ are supremum, infimum of x and y on the order $\leqslant$ respectively.

Next we shall prove the distributivity. A simple condition by P. Lorenzen that a lattice be distributive is that

(J) $x\wedge y\leqslant z, x\leqslant y\vee z$ imply $x\leqslant z$.

(For detail, see H. B. Curry [1], p. 137.) The first two conditions are respectively expressed by $y*(\sim x)\leqslant z$, $\sim z*y\leqslant\sim x$.

From $y*(\sim x)\leqslant z$, we have $x*(\sim y)\leqslant z$. Hence $x*z\leqslant\sim y$.

From $\sim z*y\leqslant\sim x$, we have $\sim z*\sim x\leqslant y$. This shows $x*z\leqslant y$.

Since $\boldsymbol{M}$ is a B-algebra, by axiom 3, we have

$$x*(x*(\sim y))\leqslant x*y.$$

If we substitute $x*z$ for x, then

$$(x*z)*((x*z)*(\sim y))\leqslant(x*z)*y.$$

As shown above, $(x*z)*(\sim y)=(x*z)*y=0$, hence we have $x*z=0$, i.e. $x\leqslant z$.

Therefore we complete the proof of our statement. Conversely we shall check that a Boolean algebra $B=\langle X, 0, 1, \vee, \wedge, \sim\rangle$ with unit satisfies axioms $1\sim5$, where $\sim x$ is the complement of x. $\boldsymbol{B}$ is defined by conditions (A)$\sim$(J). We put $x*y=\sim y\wedge x=\sim(y\vee\sim x)$.

Then $x*y=0\Longleftrightarrow\sim y\wedge x=0\Longleftrightarrow x\leqslant y$.

Next we have $x*y=\sim y\wedge x\leqslant x$, which shows axiom 1.

Further $(x*z)*(y*z)=(\sim z\wedge x)*(\sim z\wedge y)=\sim(\sim z\wedge y)\wedge(\sim z\wedge x)=(z\wedge\sim y)\wedge(\sim z\wedge x)=\sim y\wedge\sim z\wedge x=\sim z\wedge(\sim y\wedge x)=\sim z\wedge(x*y)=(x*y)*=z$. This shows axiom 2. On axiom 3, $x*(y*(\sim x))=x*(x\wedge y)=\sim(x\wedge y)\wedge x=(\sim x\vee\sim y)\wedge x=\sim y\wedge x=x\wedge(\sim y)=(\sim y)*(\sim x)$. Axioms 4 and 5 follow from the definition of a Boolean algebra.

Hence we have the following

Theorem. A Boolean algebra is characterized by axioms $1\sim5$.

Now we shall prove that the concepts of B-algebra (see [4] and [5]) and Boolean algebra are equivalent. If an algebra $\boldsymbol{M}$ satisfies axioms $1\sim5$, then by (9) and (16), we have $x*y=(\sim y)*(\sim x)$. Therefore $\boldsymbol{M}$ is a B-algebra. Conversely, the axioms of B-algebra (see [5], axioms $L1\sim L4$) imply $x*y=(\sim y)*(\sim x)$ and $x*(x*(\sim y))\leqslant x*y$ (see [4] and [5]). Hence $x*(y*(\sim x))=x*(x*(\sim y))\leqslant x*y=(\sim y)*(\sim x)$, which is axiom 3. Therefore we can formulate the following

Theorem. The concept of B-algebra coincides with the concept of Boolean algebra.

References

[1] H. B. Curry: Foundations of mathematical logic. New York (1963).
[2] P. R. Halmos: Lectures on Boolean algebras. Princeton (1963).
[3] K. Iséki: On axiom systems of propositional calculi. IV. Proc. Japan Acad.,

41, 575-577 (1965).
[4] ——: Algebraic formulations of propositional calculi. Proc. Japan Acad., **41**, 803-807 (1965).
[5] ——: Axiom systems of B-algebra. Proc. Japan Acad., **41**, 808-811 (1965).