194. On Near-algebras of Mappings on Banach Spaces

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1. A real vector space $\mathcal{A}$ is called *a near-algebra* if, for any pair of elements f and g in $\mathcal{A}$, the product fg is defined and satisfies the following two conditions:

(1) $(fg)h=f(gh)$; (2) $(f+g)h=fh+gh$.

The left distributive law: $h(f+g)=hf+hg$ is not assumed. Therefore, a near-algebra is a near-ring which has firstly been defined by [4, pp. 71–74].

A subset $\boldsymbol{I}$ of a near-algebra $\mathcal{A}$ is called *an ideal* if (1) $\boldsymbol{I}$ is a linear subset of $\mathcal{A}$; (2) $f\in\boldsymbol{I}$, $g\in\mathcal{A}$ imply $fg, gf\in\boldsymbol{I}$.

Let E be a real Banach space. Let f and g are mappings of E into E. We define the linear combination $\alpha f+\beta g$ (α and β are real numbers) by

$$(\alpha f+\beta g)(x)=\alpha f(x)+\beta g(x) \quad \text{for every} \quad x\in E,$$

and the product fg by

$$(fg)(x)=f[g(x)] \qquad \text{for every} \quad x\in E.$$

Let $\mathcal{A}$ be a near-algebra whose elements are mappings of E into E. If $\mathcal{A}$ contains the Banach algebra $\boldsymbol{L}$ of all bounded linear mappings of E into E (the norm of $\boldsymbol{L}$ is $\|l\|=\sup_{\|x\|\leq 1}\|l(x)\|$ for $l\in\boldsymbol{L}$), then, for any ideal $\boldsymbol{I}$ of $\mathcal{A}$, the set

$$\boldsymbol{I}(\boldsymbol{L})=\boldsymbol{I}\cap\boldsymbol{L}$$

is an ideal of the Banach algebra $\boldsymbol{L}$.

Examples. Let $\boldsymbol{B}$ be the near-algebra of all bounded (i.e., transforms every bounded set into a bounded set) and continuous mappings. The following subsets are ideals (cf. [3]).

1. The set $\boldsymbol{I}(E)$ of all constant mappings, in other words, $\boldsymbol{I}(E)$ is the set of all mappings $C_a(a\in E)$ such that $C_a(x)=a$ for every $x\in E$.

2. The set $\boldsymbol{C}$ of all compact (i.e., transfroms every bounded set into a compact set) and continuous mappings.

3. The set $\boldsymbol{EB}$ of all entirely bounded (i.e., transforms the space E into a bounded set) and continuous mappings.

It is obvious that $\boldsymbol{B}$ contains $\boldsymbol{L}$ and

$\boldsymbol{I}(E)\cap\boldsymbol{L}=\boldsymbol{EB}\cap\boldsymbol{L}=0$ (zero-ideal of $\boldsymbol{L}$);

$\boldsymbol{C}\cap\boldsymbol{L}=\boldsymbol{CL}$ (the set of all compact continuous linear mappings on E).

2. A mapping f of E into E is said to be (*Fréchet*) *differ-*

entiable at $a\in E$ if there exists a mapping $l\in \boldsymbol{L}$ such that

$$f(a+x)-f(a)=l(x)+r(a, x) \quad \text{for every} \quad x\in E$$

where $\lim\limits_{||x||\to 0} \dfrac{r(a, x)}{||x||}=0$.

This mapping l depends an a and denoted by $f'(a)$.

It is evident that every $l\in \boldsymbol{L}$ is differentiable at every point of E and $l'(a)=l$ for every $a\in E$.

Let $\mathcal{A}$ be a near-algebra whose elements are differentiable at every point of E. An ideal $\boldsymbol{I}$ of $\mathcal{A}$ is called *a d-ideal* if it satisfies the following conditions:

$f\in \boldsymbol{I}$ if and only if $f'(x)\in \boldsymbol{I}$ for every $x\in E$. When $\mathcal{A}=\boldsymbol{L}$, every ideal of $\mathcal{A}$ is obviously a d-ideal.

Let us assume that $\mathcal{A}$ contains $\boldsymbol{L}$. Then, the following lemmas can be proved easily.

Lemma 1. *Let $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ be d-ideals of $\mathcal{A}$. Then,*

(1) $\boldsymbol{I}_1(\boldsymbol{L})=\boldsymbol{I}_2(\boldsymbol{L})$ *implies* $\boldsymbol{I}_1=\boldsymbol{I}_2$;

(2) $\boldsymbol{I}_1(\boldsymbol{L})\subsetneqq\boldsymbol{I}_2(\boldsymbol{L})$ *implies* $\boldsymbol{I}_1\subsetneqq\boldsymbol{I}_2$.

Lemma 2. *If $\boldsymbol{I}$ is a d-ideal such that $\boldsymbol{L}\subset\boldsymbol{I}$, then $\boldsymbol{I}=\mathcal{A}$.*

3. A mapping f of E into E is said to be *compactly differentiable* if it is differentiable at every point and the mapping f' of E into $\boldsymbol{L}$ is compact. The set of all compactly differentiable mappings of E into E is denoted by $\boldsymbol{D}_c$.

Lemma 3. (1) $\boldsymbol{L}\subset\boldsymbol{D}_c$; (2) $\boldsymbol{D}_c\subset\boldsymbol{B}$; (3) $\boldsymbol{D}_c$ *is a near-algebra.*

Proof. (1) Since, for $l\in\boldsymbol{L}$, $l'(x)=l$ for every $x\in E$, the image $l'(E)$ is a one-point-set of $\boldsymbol{L}$.

(2) For the set $Br=\{x\in E\mid ||x||\leqq r\}$, since f' is a bounded mapping of E into $\boldsymbol{L}$, there exists $\alpha>0$ such that

$$||f'(x)||\leqq\alpha \qquad \text{for every} \qquad x\in Br.$$

Then, by [2, p. 37, Lemma 3.3], we have

$$||f(x)-f(0)||\leqq\alpha r \qquad \text{for every} \qquad x\in Br.$$

Therefore, f transforms every bounded set into a bounded set. The continuity of f follows from the differentiability.

(3) We have only to prove that $fg\in\boldsymbol{D}_c$ if f and g are in $\boldsymbol{D}_c$. Since $(fg)'(x)=f'[g(x)]g'(x)$ for every $x\in E$ (cf. [2, p. 41]), for any sequence $\{y_n\}\subset(fg)'(B_r)$, there exists a sequence $\{x_n\}$ such that

$$y_n=f'[g(x_n)]g'(x_n) \quad \text{and} \quad x_n\in B_r \ (n=1, 2, \cdots).$$

Since g' is compact, there exists a subsequence $\{x_k\}$ of $\{x_n\}$ such that $g'(x_k)\to l_1\in\boldsymbol{L}$ in $\boldsymbol{L}$. Since the sequence $\{g(x_k)\}$ is bounded by (2) above, there exists a subsequence $\{x_i\}$ of $\{x_k\}$ such that $f'[g(x_i)]\to l_2\in\boldsymbol{L}$ in $\boldsymbol{L}$. Therefore, in the Banach algebra $\boldsymbol{L}$, the sequence $\{f'[g(x_i)]g'(x_i)\}$ converges to $l_2l_1\in\boldsymbol{L}$.

In this near-algebra $\boldsymbol{D}_c$, we firstly characterize the set $\boldsymbol{I}(E)$.

Theorem 1. (1) *$I(E)$ is a d-ideal of D_c;*

(2) *For any non-zero ideal I of D_c, $I(E)\subset I$.*

Proof. (1) Since $C'_a(x)=0$ for every $x\in E$, $I(E)\subset D_c$. Moreover, since

$$\alpha C_a+\beta C_b=C_{\alpha a+\beta b} \quad \text{and} \quad C_a f=C_a \quad \text{for every} \quad f\in D_c,$$

$I(E)$ is obviously an ideal of D_c. Therefore, we have only to prove that $I(E)$ is a d-ideal.

(i) If $f\in I(E)$, then $f'(x)=0\in I(E)$ for every $x\in E$.

(ii) If $f'(x)\in I(E)$ for every $x\in E$, then $f'(x)=0$ for every $x\in E$. In fact, if $f'(a)\neq 0$, for an element b such that $f'(a)(b)\neq 0$, we have $f'(a)(b)\neq f'(a)(2b)$, which means that $f'(a)\notin I(E)$. Therefore, $f\in I(E)$.

(2) Let $f\in I$ be an arbitrary non-zero element. Then, $C_a=C_a f\in I$ for every $a\in E$, i.e., $I(E)\subset I$.

Secondly, we characterize the set $I(C)=C\cap D_c$ of all compact mappings which are compactly differentiable at every point.

Theorem 3. (1) *$I(C)$ is a proper d-ideal of D_c;*

(2) *For any d-ideal I of D_c, if $CL\subset I$, then $I(C)\subset I$.*

When E is a separable Hilbert space,

(3) *For any proper d-ideal I of D_c, we have $I\subset I(C)$.*

Proof. (1) Since C and D_c are linear, $I(C)$ is obviously linear. Assume that $f\in I(C)$ and $g\in D_c$. By Lemma 3, fg and gf belong to D_c. Moreover, for a bounded set B, $g(B)$ is bounded by Lemma 3 and $f(B)$ is contained in a compact set. Therefore, $f[g(B)]$ and $g[f(B)]$ are contained in compact sets, which means that fg and gf belong to $I(C)$. Finally, we prove that $I(C)$ is a d-ideal. If $f\in I(C)$, then $f'(x)\in L$ is compact and continuous for every $x\in E$ (cf. [2, p. 51, Theorem 4.7]), which means that $f'(x)\in I(C)$ for every $x\in E$. Conversely, if $f'(x)\in I(C)$ for every $x\in E$, then by [2, p. 51, Theorem 4.8], we have $f\in I(C)$.

(2) For any $f\in I(C)$, $f'(x)\in CL$ for every $x\in E$, hence it follows that $f'(x)\in I$ for every $x\in E$. Since I is a d-ideal, $f\in I$.

(3) Let I be a proper d-ideal of D_c. $I(L)=I\cap L$ is an ideal of L. Then, as Calkin has proved in [1], we have either $I(L)=L$ or $I(L)\subset CL$. If $I(L)=L$, then $L\subset I$. By Lemma 2, we have $I=D_c$, which shows that I is not proper. If $I(L)\subset CL$, then

$$I\cap L\subset CL=C\cap L=I(C)\cap L.$$

Therefore, by Lemma 1, we have $I\subset I(C)$.

References

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