191. On Complete Degrees

By Ken HIROSE

Department of the Foundations of Mathematical Sciences,
Tokyo University of Education

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In his paper [2], R. M. Friedberg proved that a degree of recursive unsolvability $\boldsymbol{a}$ is complete if and only if $\boldsymbol{a} \geqq \boldsymbol{0}'$. The aim of this note is to prove the following: *for each degree* $\boldsymbol{a}$, *there exist infinitely many independent degrees* $\boldsymbol{b}_0, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n, \cdots$ *whose completion are* $\boldsymbol{a}$ *if and only if* $\boldsymbol{a} \geqq \boldsymbol{0}'$. This will be shown as a corollary to the following.

Theorem. *For each degree* $\boldsymbol{a}$, *there exist infinitely many degrees* $\boldsymbol{b}_0, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n, \cdots$ *such that*:

(1) $\boldsymbol{b}_0, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n, \cdots$ *are independent*,

(2) $\boldsymbol{b}_i' = \boldsymbol{b}_i \cup \boldsymbol{0}' = \boldsymbol{a} \cup \boldsymbol{0}'$ *for* $i = 0, 1, \cdots, n, \cdots$

Let $\alpha(x)$ be a function of degree $\boldsymbol{a}$. We shall construct a function $\lambda xi\beta(x, i)$ such that $\lambda x\beta(x, i)(=\beta_i(x))$ is not recursive in $\lambda xz\beta(x, z+sg((z+1)\dot{-}i))(=\beta^i(x, z))$ and satisfies (2). And let $\boldsymbol{b}_i$ be the degree of $\beta_i(x)$. As in [1], $\lambda xi\beta(x, i)$ is constructed by defining inductively functions $\psi(s)$ and $\nu(s)$ such that

$$\beta(x, i) = (\psi(s))_{x,i} \quad \text{for each} \quad x < \nu(s) \quad \text{and each} \quad i < \nu(s).$$

1. First, we shall define a recursive predicate $\text{comp}(s_1, s_2)$ and function $\phi(e, v)$ of degree $\boldsymbol{0}'$ as follows:

$$\text{comp}(s_1, s_2) \equiv (u_1)_{u_1 < 1h(s_1)} (u_2)_{u_2 < 1h(s_2)} (u_3)_{u_3 < \min(1h(s_1), 1h(s_2))} [(s_1)_{u_1} \neq 0 \ \& \ (s_2)_{u_2} \neq 0 \ \& \ (s_1)_{u_3} = (s_2)_{u_3}],$$

$$\phi(e, v) = \begin{cases} \mu s(T_1^1(s, e, e) \ \& \ \text{comp}(s, v)) \\ \qquad \text{if } (Es)(T_1^1(s, e, e) \ \& \ \text{comp}(s, v)), \\ 0 \qquad \text{otherwise}, \end{cases}$$

where $T_1^1(\prod_{u<y} p_u^{f(u)+1}, e, x) \equiv T_1^f(e, x, y)$.

Now, we shall define the functions $\nu(s)$ and $\psi(s)$ simultaneously by the induction on the number s, and put $\beta(x, i) = (\psi(s))_{x,i}$ for each $x < \nu(s)$ and each $i < \nu(s)$.

Stage $s = 0$.

$\nu(0) = 0$,

$\psi(0) = 1$.

Stage $s+1$.

Case 1: $(Ey)T_1^2(\widetilde{\beta}^{(s)_0}(y, y), (s)_1, \nu(s), y)$.

This means that

$(Ey)(Eb)[b \neq 0 \ \& \ (i)_{i<y}((b)_i \neq 0) \ \& \ (i)_{i<y}(j)_{j<y}((b)_{i,j} < 2) \ \&$

$(i)_{i<\nu(s)}(j)_{j<\nu(s)\dot{-}sg(\nu(s)\dot{-}(s)_0)}((b)_{i,j}=(\psi(s))_{i,j+sg((j+1)\dot{-}(s)_0)})$ & $T_1^2(b,(s)_1,\nu(s),y)]$.
If we put $x=2^y\cdot 3^b$, this is in the form

$$(Ex)R(s,\nu(s),\psi(s),x)$$

where R is recursive predicate.

We define the functions as follows:

$$\xi(s)=\mu x R(s,\nu(s),\psi(s),x).$$
$$\nu(s+1)=\max(\nu(s)+2,(s)_0+1,\xi(s)+1)$$

and

$$\begin{aligned}\psi(s+1)=\mu t[&t\neq 0 \ \& \ (i)_{i<\nu(s+1)}((t)_i\neq 0) \ \& \\ &(i)_{i<\nu(s)}(j)_{j<\nu(s)}((t)_{i,j}=(\psi(s))_{i,j}) \ \& \\ &(i)_{i<(\xi(s))_0}(j)_{j<(\xi(s))_0}((t)_{i,j+sg((j+1)\dot{-}(s)_0)}=((\xi(s))_1)_{i,j}) \ \& \\ &((t)_{\nu(s),(s)_0}=\overline{sg}(U((\xi(s))_0))) \ \& \\ &(j)_{j<\nu(s+1)}((t)_{\nu(s)+1,j}=\alpha(s))].\end{aligned}$$

Case 2: otherwise.

$$\nu(s+1)=\nu(s)+2,$$
$$\psi(s+1)=\psi(s)\cdot p_{\nu(s)}\cdot p_{\nu(s)+1}\exp\Big(\prod_{j<\nu(s+1)}p_j^{\alpha(s)}\Big).$$

We set

$$\beta(x,i)=(\psi(s+1))_{x,i}$$

for each $x<\nu(s+1)$ and each $i<\nu(s+1)$.

2. We shall prove (1). For each i, we consider numbers s such that $(s)_0=i$. If Case 1 holds, then

$$\beta(\nu(s),i)=(\psi(s+1))_{\nu(s),(s)_0}=\overline{sg}(U((\xi(s))_0)).$$

Thus, for each $e=(s)_1$,

$$\beta_i(\nu(s))\neq U(\mu y T_1^2(\tilde{\beta}^i(y,y),e,\nu(s),y)).$$

If Case 2 holds, then

$$(\overline{Ey})T_1^2(\tilde{\beta}^i(y,y),(s)_1,\nu(s),y).$$

Then, we obtain

$$\beta_i(x) \text{ is not recursive in } \beta^i(x) \text{ for all } i.$$

That is, $\boldsymbol{b}_0,\boldsymbol{b}_1,\cdots,\boldsymbol{b}_n,\cdots$ are independent.

3. Now, we shall prove (2). By the definition of ϕ and ψ, it is easily see that

(i) $(e)(i)[(Ey)T_1^1(\bar{\beta}_i(y),e,e)\to\phi(e,\prod_{j<\nu(e+1)}p_j^{(\psi(e))_{j,i+1}})\neq 0]$,

(ii) $(e)(i)[\phi(e,\prod_{j<\nu(e+1)}p_j^{(\psi(e))_{j,i+1}})\neq 0$
$\to T_1^{\beta_i}(e,e,lh(\phi(e,\prod_{j<\nu(e+1)}p_j^{(\psi(e))_{j,i+1}})))]$.

From (i) and (ii), we obtain

$$(e)(i)[(Ey)T_1^1(\bar{\beta}_i(y),e,e)\equiv\phi(e,\prod_{j<\nu(e+1)}p_j^{(\psi(e))_{j,i+1}})\neq 0].$$

Therefore,

(iii) $\boldsymbol{b}_i'\leqq\boldsymbol{a}\cup\boldsymbol{0}'$ for each i,

because ψ is recursive in a function of degree $\boldsymbol{a}\cup\boldsymbol{0}'$.

On the other hand, we have

$$(x)(i)(u)_{u>\max(x+1,i)}[\alpha(x)=(\psi(u+1))_{\nu(x)+1,i}=\beta(\nu(x)+1,\, i)$$
$$=\beta_i(\nu(x)+1)].$$

Since ν is recursive in a function of degree $\mathbf{0}'$,

(iv) $\quad \boldsymbol{a}\leqq \boldsymbol{b}_i \cup \mathbf{0}' \quad$ for each i,

which implies

(v) $\quad \boldsymbol{a}\cup \mathbf{0}' \leqq \boldsymbol{b}_i \cup \mathbf{0}' \quad$ for each i.

Since $\boldsymbol{b}_i' \geqq \mathbf{0}'$, we have

(vi) $\quad \boldsymbol{b}_i \cup \mathbf{0}' \leqq \boldsymbol{b}_i' \quad$ for each i.

Thus, by (iii), (v), and (vi), we obtain,

$$\boldsymbol{b}_i' \leqq \boldsymbol{a}\cup \mathbf{0}' \leqq \boldsymbol{b}_i \cup \mathbf{0}' \leqq \boldsymbol{b}_i',$$

that is,

$$\boldsymbol{b}_i' = \boldsymbol{a}\cup \mathbf{0}' = \boldsymbol{b}_i \cup \mathbf{0}' \quad \text{for each } i.$$

4. **Corollary.** *For each degree* $\boldsymbol{a}$, *there exist infinitely many independent degrees* $\boldsymbol{b}_0, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n, \cdots$ *whose completion are* $\boldsymbol{a}$ *if and only if* $\boldsymbol{a}\geqq \mathbf{0}'$.

Proof. Apply our theorem with $\boldsymbol{a}\geqq \mathbf{0}'$. Then we obtain infinitely many independent degrees $\boldsymbol{b}_0, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n, \cdots$ such that $\boldsymbol{b}_i' = \boldsymbol{a}\cup \mathbf{0}'$ for each i. Since $\boldsymbol{a}\geqq \mathbf{0}'$, we have

$$\boldsymbol{b}_i' = \boldsymbol{a} \quad \text{for each } i.$$

References

[1] S. C. Kleene and E. L. Post: The upper semi-lattice of degrees of recursive unsolvability. Ann. of Math., **59** (1954).

[2] R. M. Friedberg: A criterion for completeness of degrees of unsolvability. J. Symb. Log., **22** (1957).

[3] S. C. Kleene: Introduction to Metamathematics. New York, Tronto, Amsterdam, and Gröningen (1952).