7. An Algebra Related with a Propositional Calculus

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(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1966)

In this note, we shall consider a new algebra induced by the *BCI*-system of propositional calculus by C. A. Meredith quoted into A. N. Prior, Formal Logic ([4], p. 316).[1]

Unfortunately, we can not find the details on the *BCI* and *BCK*-systems in literatures. For the completeness, giving its detail, we shall develop our consideration.

If we take the *BCI*-system or the weak positive implicational calculus by A. Church, these systems are given by the following axioms.

BCI-system: *CCpqCCqrCpr*, *CpCCpqq*, and *Cpp*,

WPI-system: *CCpCpqCpq*, *CCqrCCpqCpr*, *CCpCqrCqCpr*, and *Cpp*.

In these systems, we can not deduce an important thesis: *CpCqp*. From an attempt of algebraic formulations, we have a quite different situation from our former discussions (see [1], [2]).

Let $M=\langle X, 0, *\rangle$ be an abstract algebra consisting of a set X with an element 0 and a binary operation $*$. If M satisfies the following conditions *BCI* $1\sim 5$, it is called a *BCI-algebra*.

BCI 1 $(x*y)*(x*z)\leqslant z*y$,

BCI 2 $x*(x*y)\leqslant y$,

BCI 3 $x\leqslant x$,

BCI 4 $x\leqslant y$, $y\leqslant x$ imply $x=y$,

BCI 5 $x\leqslant 0$ implies $x=0$,

where $x\leqslant y$ means $x*y=0$.

Here we do not assume $0*x=0$, i.e. $0\leqslant x$. This is an essential part and differs from axiom systems formulated in our previous notes [1], [2]. *BCI* 5 shows that $x*0=0$ implies $x=0$. And we have $0*x=0*0=0$. Hence if $x*0=0*x=0$, then $x=0$.

From the first axiom, we have the following important results:

(1) $x\leqslant y$ implies $z*y\leqslant z*x$. $x\leqslant y$, $y\leqslant z$ imply $x\leqslant z$.

By (1), if $x=y$, $y=z$, then $x=z$.

Theorem 1. *The second axiom in BCI-algebra is replaced by*

(2) $(x*y)*z\leqslant(x*z)*y$.

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1) In my seminar on mathematical logic, Mr. Shôtarô Tanaka announced forming rules to produce a single axiom from several axioms of *CN*-types in propositional calculi. His results are based on so called *BCI*, *BCK*-systems introduced by C. A. Meredith (see A. N. Prior [4], p. 316).

Proof. Assume axioms 1, 3 and the condition (2), then

$$(x*(x*y))*y \leqslant (x*y)*(x*y)=0.$$

By *BCI* 5, we have $(x*(x*y))*y=0$, i.e. $x*(x*y)\leqslant y$.

Conversely, we shall show that axioms of *BCI*-algebra imply (2). By (1), we have

(3) $u*(z*y)\leqslant u*((x*y)*(x*z))$.

We substitute $x*u$ for x, $x*z$ for z, $((x*u)*y)*(z*u)$ for u in (3), and use (3), then we have

$$((x*u)*y)*(z*u))*((x*z)*y)$$
$$\leqslant (((x*u)*y)*(z*u))*(((x*u)*y)*((x*u)*(x*z)))=0.$$

Hence we have

(4) $((x*u)*y)*(y*u)\leqslant (x*z)*y$,

which is a thesis in J. Lukasiewicz [2]. In this formula, let $u=z$, $z=x*y$, then we have the following formula

$$((x*z)*y)*((x*y)*z)\leqslant (x*(x*y))*y.$$

The right side is equal to 0 by *BCI* 2. Therefore we have $(x*z)*y\leqslant (x*y)*z$, which means the formula (2). We complete the proof of Theorem 1.

Remark 1. Under the formula (2), $(x*y)*(x*z)\leqslant z*y$ and $(x*y)*(z*y)\leqslant x*z$ are equivalent.

Remark 2. Let $(x*y)*z=0$, i.e. $x*y\leqslant z$, then by (1), we have $x*z\leqslant x*(x*y)\leqslant y$. Hence if $x*y\leqslant z$, then $x*z\leqslant y$.

Remark 3. Formula (2) is written in form of

$$(x*y)*z=(x*z)*y$$

by *BCI* 4.

Theorem 2. *In a BCI-algebra, we have the following formulas:*

(5) $((x*y)*z)*(u*z)\leqslant (x*u)*y$,

(6) $((x*y)*z)*((x*u)*y)\leqslant u*z$,

(7) $(x*y)*(z*u)\leqslant x*(z*(u*y))$,

(8) $(x*y)*(x*(z*(u*y)))\leqslant z*u$.

Proof. By the formula

(9) $(x*y)*(z*y)\leqslant x*z$

and (2), we have

$$((x*y)*z)*(u*z)\leqslant (x*y)*u\leqslant (x*u)*y,$$

which shows formula (5). Formula (6) is obtained by (2) and (5). By Remark 3, we have $(x*y)*(z*u)=(x*(z*u))*y$ and by using *BCI* 1 and 2,

$$(x*(z*u))*(x*(z*(u*y)))\leqslant (z*(u*y))*(z*u)\leqslant u*(u*y)\leqslant y.$$

Therefore

$$(x*(z*u))*y\leqslant x*(z*(u*y)),$$

and we have

$$(x*y)*(z*u)\leqslant x*(z*(u*y)),$$

which is formula (7). Further, by (2) and (7), we have formula (8). Therefore the proof is complete.

Formulas (5), (6), (7), and (8) are found in [4].

If *BCI* 5 is replaced by $0*x=0$, i.e.

BCI 6. $\qquad 0\leqslant x$ for every $x\in X$,

then, by (2) and *BCI* 6, we have

$$(x*y)*x\leqslant(x*x)*y=0$$

for every $y\in X$. Hence we have $x*y\leqslant x$. Conversely, if $x*y\leqslant x$ and (2) hold, then $0=x*x\leqslant y$ for every $y\in X$.

An algebra M satisfying *BCI* 1~4 and *BCI* 6 corresponds to the *BCK* system on propositional calculus introduced by C. A. Meredith. If the axiom *Cpp* in the *BCI* system of propositional calculus is replaced by *CpCqp*, then we obtain the *BCK* system of propositional calculus (see A. N. Prior [4]). Therefore M is called *BCK-algebra.*

Next we shall consider formulas to characterize two algebras mentioned above.

Consider an algebra satisfying *BCI* 3~5 and (5), then it is a *BCI*-algebra.

To prove it, we first show (5) implies (6) under the condition *BCI* 3~5. Assume that formula (5), if $x*u\leqslant y$, then $(x*y)*z\leqslant u*z$. Put $u=z$, then we have $x*y\leqslant z$, if $x*z\leqslant y$. Hence *BCI* 3~5 and (5) imply that if $x*y\leqslant z$, then $x*z\leqslant y$. Hence *BCI* 3~5 and (5) imply (6). In formula (6), let $u=z$, then $(x*y)*z\leqslant(x*z)*y$. Hence by *BCI* 4, we have

(10) $\quad (x*y)*z=(x*z)*y.$

In formula (5), let $y=x*u$, then

$$(x*(x*u))*z\leqslant u*z.$$

Applying (10), then we have

$$(x*y)*(x*z)\leqslant z*y,$$

which is *BCI* 1. Further, in (5), put $y=x*x, u=z=y$, then

$$((x*(x*y)*y)*(y*y)\leqslant(x*y)*(x*y)=0.$$

Since $y*y=0$, we have $x*(x*y)*y=0$, which is *BCI* 2.

Next, assume (6) and *BCI* 3~5. As shown above, (6) implies (10). Hence under these conditions, we have (5). Therefore we have the following

Theorem 3. *A BCI-algebra is characterized by BCI* 3~5 *and* (5) (*or* (6)),

and

Theorem 4. *A BCK-algebra is characterized by BCI* 3, 4, 6, *and* (5) (*or* (6)).

Next consider *BCI* 3~5, and (7). In formula (7), put $x=(x*y)*(z*u)$, $y=u*y$, $z=x$, and $u=z$, then we have

$$(((x*y)*(z*u))*(u*y))*(x*z)$$
$$\leqslant((x*y)*(z*u))*(x*(z*(u*y)))=0$$

by (7). Hence $((x*y)*(z*u))*(u*y)\leqslant x*z$. Further in this formula, let $x*z=0$, $u=y$, then we have $((x*y)*(z*u))*0=0$, and we have $x*y\leqslant z*y$. Hence if $x\leqslant z$, then $x*y\leqslant z*y$. Therefore $x\leqslant y$, $y\leqslant z$ imply $x\leqslant z$. To deduce *BCI* 1, we again use formula (7). We substitute x for z, z for u in (7), then

$$(x*y)*(x*z)\leqslant x*(x*(z*y)).$$

On the other hand, in (7), put $x=z*(u*y)$, then

$$(((z*(u*y))*y)*(z*u)\leqslant(z*(u*y))*(z*(u*y))=0,$$

and we have $(z*(u*y))*y\leqslant z*u$. Let $u=z$ in this formula, then we have $x*(x*y)\leqslant y$, which is *BCI* 2. This implies $x*(x*(z*y))\leqslant z*y$. Therefore we have the following formula

$$(x*y)*(x*z)\leqslant x*(x*(z*y))\leqslant z*y,$$

which is *BCI* 1. Hence we have the following

Theorem 5. *A BCI-algebra is characterized by BCI* 3~5 *and* (7). *A BCK-algebra is characterized by BCI* 3, 4, 6, *and* (7).

Remark 4. S. Tanaka has shown (in an oral communication) that the *BCI*, *BCK* algebras are characterized by using formula (8). The detail will be published.

Remark 5. Roughly speaking, 0 is the least element in the *BCI*-algebra, and a minimal element in the *BCK*-algebra. An algebra having 0 as the least element is *Sobociński algebra* formulated by three value calculus (see [5]).

References

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