5. *On Axiom Systems of Propositional Calculi. XIV*

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(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1966)

All this while, we considered axiom systems of classical propositional calculus. In this note, we shall treat the implicational (propositional) calculus originated by A. Tarski and P. Bernays. The fundamental axioms of the implicational calculus are given by

1 $CpCqp$,

2 $CCpqCCqrCpr$,

3 $CCCpqpp$

and two usual rules of inference.

In his Formal Logic [2], A. N. Prior has proved several theses from the above axioms. We shall follow an algebraic formulation of implicational calculus to prove some theses (for the detail [1]). To do so, we consider an algebra $\boldsymbol{M}=\langle X, 0, *\rangle$ satisfying the following conditions:

$I\,1$ $x*y\leqslant x$,

$I\,2$ $(x*y)*(x*z)\leqslant z*y$,

$I\,3$ $x\leqslant x*(y*x)$,

$I\,4$ $0\leqslant x$,

$I\,5$ $x*y=0$ if and only if $x\leqslant y$.

If $x\leqslant y$ and $y\leqslant x$, then we define $x=y$. The algebra $\boldsymbol{M}$ is called an *I-algebra*.

First we shall show some simple fundamental lemmas.

(1) $0*x=0$.

(2) $x*x=0$, i.e. $x\leqslant x$.

(3) $x*y=0, y*z=0$ imply $x*z=0$, i.e. if $x\leqslant y, y\leqslant z$, then $x\leqslant z$.

(4) $x\leqslant y$ implies $z*y\leqslant z*x$.

To prove (2), put $z=y*z$ in $I\,2$, then we have

$$(x*y)*(x*(y*z))\leqslant(y*z)*y=0.$$

Hence we have

(5) $x*y\leqslant x*(y*z)$.

Next, $y=x, z=y*x$ in (5), then $x*x\leqslant x*(x*(y*x))=0$ by $I\,3$. Hence $x*x=0$, i.e. $x\leqslant x$. By $I\,2$, $z*y=0, x*z=0$ imply $x*y=0$. Hence $x\leqslant z, z\leqslant y$ imply $x\leqslant y$, which means (3). (4) follows from $I\,2$, putting $z*y=0$.

From $I\,1, I\,3$, we have

(6) $x=x*(y*x)$.

By (3), we have $(x*y)*z\leqslant x*y\leqslant x$. Hence

(7) $(x*y)*z\leqslant x$.

In (6), put $y=0$, then

(8) $x*0=x$.

Next we shall prove a commutative law:

(9) $(x*y)*z=(x*z)*y$.

The proof is similar with A. N. Prior [2], so it is not new. In $I\,2$, put $x=z*y$, $y=(z*y)*y$, $z=(z*y)*(z*(z*y))$, then

$$((z*y)*((z*y)*y))*((z*y)*((z*y)*(z*(z*y))))$$
$$\leqslant((z*y)*(z*(z*y)))*((z*y)*y).$$

The right side is 0 by $I\,2$ and the second term of the left side is 0 by $I\,3$. Therefore we have $x*y\leqslant(x*y)*y$. By $I\,1$, we have $(x*y)*y\leqslant x*y$, hence

(10) $x*y=(x*y)*y$.

In $I\,2$, put $x=z*(z*y)$, $y=y*(z*y)$, and $z=(z*(z*y))*(z*y)$, then

$$((z*(z*y))*(y*(z*y)))*((z*(z*y))*((z*(z*y))*(z*y)))$$
$$\leqslant((z*(z*y))*(z*y))*(y*(z*y)).$$

The right side is 0 by $I\,2$, and the second term of the left side is 0 by (10), hence we have $z*(z*y)\leqslant y*(z*y)=y$ by (6). Therefore

(11) $x*(x*y)\leqslant y$.

By (4) and (11), we have

(12) $z*y\leqslant z*(x*(x*y))$.

Next, in $I\,1$, put $x=(x*y)*z$, $y=(x*z)*y$, and $z=(x*y)*(x*(x*z))$, then we have

$$(((x*y)*z)*((x*z)*y))*(((x*y)*z)*((x*y)*(x*(x*z))))$$
$$\leqslant((x*y)*(x*(x*z)))*((x*z)*y).$$

The right side is 0 by $I\,2$, and the second term of the left side is 0 by (12), hence $(x*y)*z\leqslant(x*z)*y$, therefore we have the commutative law (9).

By (9) and $I\,2$, we have

(13) $(x*y)*(z*y)\leqslant x*z$.

This means that $x\leqslant y$ implies $x*z\leqslant y*z$. By (13), $(x*z)*(y*z)\leqslant x*z$ implies

$$((x*z)*(y*z))*z\leqslant(x*y)*z.$$

By the commutative law (9), we have

$$((x*z)*z)*(y*z)\leqslant(x*y)*z.$$

On the other hand, by $I\,2$ we have

$$((x*y)*(x*z))*u\leqslant(z*y)*u.$$

By (9), we have

$$((x*y)*u)*(x*z)\leqslant(z*y)*u.$$

Put $x=x*z$, $y=y*z$, $z=(x*z)*z$, and $u=(x*y)*z$ in the above formula, then

$$(((x*z)*(y*z))*((x*y)*z))*((x*z)*((x*z)*z))$$
$$\leqslant(((x*z)*z)*(y*z))*((x*y)*z)=0$$

by the above formula. Further $x*z=(x*z)*z$ by (10), hence

(14) $(x*z)*(y*z)\leqslant(x*y)*z$.

From these formulas proved, we can obtain the results which are used to characterize the implicational calculus.

$x*y\leqslant x$ implies

(15) $(x*y)*z\leqslant x*z$.

Formula (13), axiom $I\,1$ and $0\leqslant u$ imply

$$((x*y)*(x*z))*u\leqslant(z*y)*u\leqslant z*y,$$

hence

(16) $((x*y)*(x*z))*u\leqslant z*y$.

By the similar way, $(x*y)*u\leqslant x*y$ implies

$$((x*y)*u)*(x*z)\leqslant(x*y)*(x*z)\leqslant z*y.$$

Further $x*u\leqslant x$ implies $(x*u)*y\leqslant x*y$, then

$$((x*u)*y)*(x*z)\leqslant(x*y)*(x*z)\leqslant z*y.$$

Therefore we have

(17) $((x*y)*u)*(x*z)\leqslant z*y$.

and

(18) $((x*u)*y)*(x*z)\leqslant z*y$.

From (6), $x*(z*x)=x$, hence

(19) $x*y\leqslant x*(z*x)$.

The calculation

$$(x*(y*z))*(x*(x*z))\leqslant(x*z)*(y*z)\leqslant x*y$$

implies

(20) $(x*(y*z))*(x*y)\leqslant x*(x*z)$.

Next we shall prove

(21) $(x*y)*(z*u)\leqslant x*(z*(u*y))$.

We have the following relations:

$$(x*y)*(x*(z*(u*y)))\leqslant(z*(u*y))*y,$$

$$((z*(u*y))*y)*(z*u)=((z*(u*y))*(z*u))*y\leqslant(u*(u*y))*y$$
$$=(u*y)*(u*y)=0.$$

Hence $(x*y)*(x*(z*(u*y)))\leqslant z*u$, which shows (21). The following calculation:

$$(x*(y*z))*(x*(y*u))\leqslant(y*u)*(y*z)\leqslant z*u$$

implies

$$(x*(y*z))*(z*u)\leqslant x*(y*u).$$

Put $y=x, z=y$, and $u=z*x$ in the above formula, then

$$(x*(x*y))*(y*(z*x))\leqslant x*(x*(z*x))=0$$

from $I\,3$. Therefore we have

(22) $x*(x*y)\leqslant y*(z*x)$.

Since z is arbitrary in (22), we have

(23) $x*(x*y)=y*(y*x)$.

In axiom $I\,2$, put $x=x*(x*y)$, $y=x*(z*y)$, and $z=y*(y*x)$, then

$$((x*(x*y))*(x*(z*y)))*((x*(x*y))*(y*(y*x)))$$
$$\leqslant(y*(y*x))*(x*(z*y)).$$

The right side of the above formula is 0 from (22), and the second term of the left side is 0 from (23), hence

(24) $x*(x*y)\leqslant x*(z*y)$.

From (24) and axiom $I\,1$, we have

(25) $(x*(x*y))*(x*u)\leqslant x*(z*y)$.

From (9), (13), (22), and $I\,1$,

$$(x*y)*(x*z)=(x*(x*z))*y\leqslant(z*(u*x))*y\leqslant z*(u*x).$$

Hence

(26) $(x*y)*(x*z)\leqslant z*(u*x)$.

From (22), we have $x*(x*z)\leqslant z*(v*x)$. By (13) and $I\,1$,

$$((x*(x*z))*y)*u\leqslant(x*(x*z))*y\leqslant(z*(v*x))*y.$$

Applying (9), we have

(27) $((x*y)*(x*z))*u\leqslant(z*y)*(v*x)$.

Further, by (9), we have

(28) $((x*y)*u)*(x*z)\leqslant(z*y)*(v*x)$.

From $I\,1$, $(y*x)*z\leqslant y*x$. By (3), we have $x*(y*x)\leqslant x*((y*x)*z)$. By $I\,3$, we have $x\leqslant x*((y*x)*z)$. Hence we have

(29) $x=x*((y*x)*z)$.

By $I\,1$, $x*y\leqslant x$. From (13) and $I\,1$ we have $(x*y)*z\leqslant x*z\leqslant x$. Further, by (13) and (22),

$$((x*y)*z)*(x*u)\leqslant x*(x*u)\leqslant u*(v*x).$$

Hence

(30) $((x*y)*z)*(x*u)\leqslant u*(v*x)$.

From (21), (24) we have

$$(x*y)*(x*z)\leqslant x*(x*(z*y))\leqslant x*(u*(z*y)),$$

hence

(31) $(x*y)*(x*z)\leqslant x*(u*(z*y))$.

Finally, we shall give a proof of M. Wajsberg condition

(32) $(((x*y)*(z*u))*(x*(y*v)))*w\leqslant(x*(y*v))*(t*x)$.

In this case, we see that the left side is 0. From (9) and (13)

$$\begin{aligned}((x*y)*(z*u))*(x*(y*v))&=((x*y)*(x*(y*v)))*(z*u)\\&\leqslant((y*v)*y)*(z*u)=0*(z*u)=0.\end{aligned}$$

Hence the left side of (32) is equal to $0*w=0$. Therefore we have (32).

References

[1] K. ISÉKI: Algebraic formulations of propositional calculi. Proc. Japan Acad., **41**, 803-807 (1965).

[2] A. N. Prior: Formal Logic. Oxford (1962).